3. Subgradient method

- subgradient method
- convergence analysis
- optimal step size when $f^\star$ is known
- alternating projections
- projected subgradient method
- optimality of subgradient method

Subgradient method

to minimize a nondifferentiable convex function f: choose x_0 and repeat

$$x_{k+1} = x_k - t_k g_k, \quad k = 0, 1, \ldots$$

g_k is any subgradient of f at x_k

Step size rules

- fixed step: t_k constant
- fixed length: $t_k \|g_k\|_2 = \|x_{k+1} - x_k\|_2$ is constant
- diminishing: $t_k \to 0$ and $\sum_{k=0}^{\infty} t_k = \infty$

Assumptions

- problem has finite optimal value $f^\star$, optimal solution $x^\star$
- f is convex with $\operatorname{dom} f = \mathbf{R}^n$
- f is Lipschitz continuous with constant $G > 0$:

$$|f(x) - f(y)| \leq G\|x - y\|_2 \qquad \text{for all } x, y$$

  this is equivalent to $\|g\|_2 \leq G$ for all x and $g \in \partial f(x)$ (see next page)

Proof.

- assume $\|g\|_2 \leq G$ for all subgradients; choose $g_y \in \partial f(y)$, $g_x \in \partial f(x)$:

$$g_x^T(x-y) \geq f(x) - f(y) \geq g_y^T(x-y)$$

  by the Cauchy–Schwarz inequality

$$G\|x-y\|_2 \geq f(x) - f(y) \geq -G\|x-y\|_2$$

- assume $\|g\|_2 > G$ for some $g \in \partial f(x)$; take $y = x + g/\|g\|_2$:

$$\begin{aligned} f(y) &\geq f(x) + g^T(y-x) \\ &= f(x) + \|g\|_2 \\ &> f(x) + G \end{aligned}$$

Analysis

- the subgradient method is not a descent method
- therefore $f_{\text{best},k} = \min_{i=0,\ldots,k} f(x_i)$ can be less than $f(x_k)$
- the key quantity in the analysis is the distance to the optimal set

Progress in one iteration

- distance to $x^\star$:

$$\begin{aligned}
\|x_{i+1} - x^\star\|_2^2 &= \left\|x_i - t_i g_i - x^\star\right\|_2^2 \\
&= \|x_i - x^\star\|_2^2 - 2t_i g_i^T (x_i - x^\star) + t_i^2 \|g_i\|_2^2 \\
&\leq \|x_i - x^\star\|_2^2 - 2t_i \left(f(x_i) - f^\star\right) + t_i^2 \|g_i\|_2^2
\end{aligned}$$

- best function value: combine inequalities for $i = 0, \ldots, k$:

$$\begin{aligned}
2\left(\sum_{i=0}^{k} t_i\right)(f_{\text{best},k} - f^\star) &\leq \|x_0 - x^\star\|_2^2 - \|x_{k+1} - x^\star\|_2^2 + \sum_{i=0}^{k} t_i^2 \|g_i\|_2^2 \\
&\leq \|x_0 - x^\star\|_2^2 + \sum_{i=0}^{k} t_i^2 \|g_i\|_2^2
\end{aligned}$$

Fixed step size and fixed step length

Fixed step size: $t_i = t$ with t constant

$$f_{\text{best},k} - f^\star \leq \frac{\|x_0 - x^\star\|_2^2}{2(k+1)t} + \frac{G^2 t}{2}$$

- does not guarantee convergence of $f_{\text{best},k}$
- for large k, $f_{\text{best},k}$ is approximately $G^2 t/2$-suboptimal

Fixed step length: $t_i = s/\|g_i\|_2$ with s constant

$$f_{\text{best},k} - f^\star \leq \frac{G\|x_0 - x^\star\|_2^2}{2(k+1)s} + \frac{Gs}{2}$$

- does not guarantee convergence of $f_{\text{best},k}$
- for large k, $f_{\text{best},k}$ is approximately $Gs/2$-suboptimal

Diminishing step size

$$t_i \to 0, \qquad \sum_{i=0}^{\infty} t_i = \infty$$

- bound on function value:

$$f_{\mathrm{best},k} - f^\star \le \frac{\|x_0 - x^\star\|_2^2}{2\sum_{i=0}^{k} t_i} + \frac{G^2 \sum_{i=0}^{k} t_i^2}{2\sum_{i=0}^{k} t_i}$$

- can show that $(\sum_{i=0}^{k} t_i^2)/(\sum_{i=0}^{k} t_i) \to 0$; hence, $f_{\mathrm{best},k}$ converges to $f^\star$
- examples of diminishing step size rules:

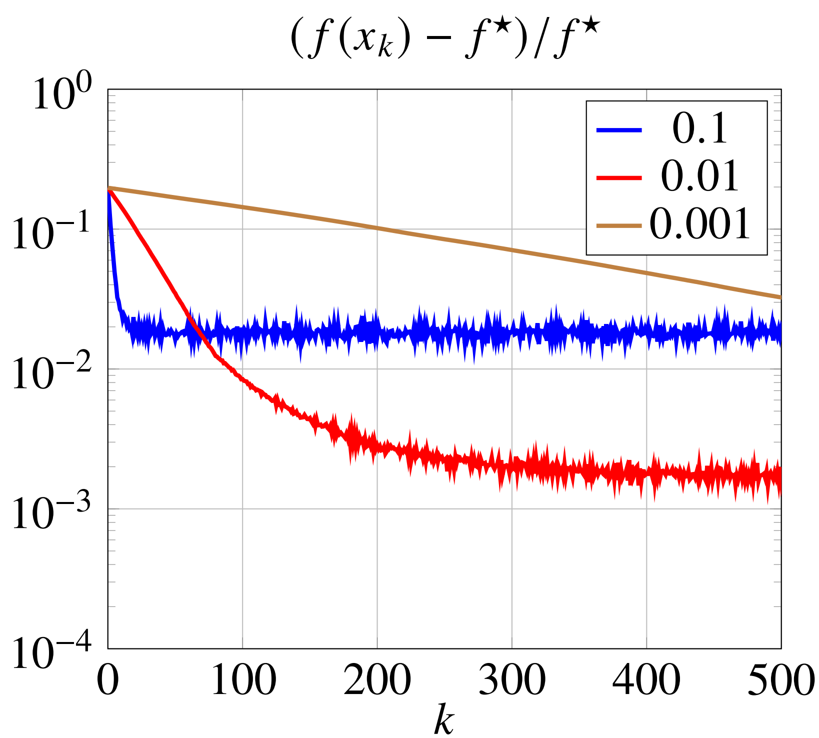
$$t_i = \frac{\tau}{i+1}, \qquad t_i = \frac{\tau}{\sqrt{i+1}}$$

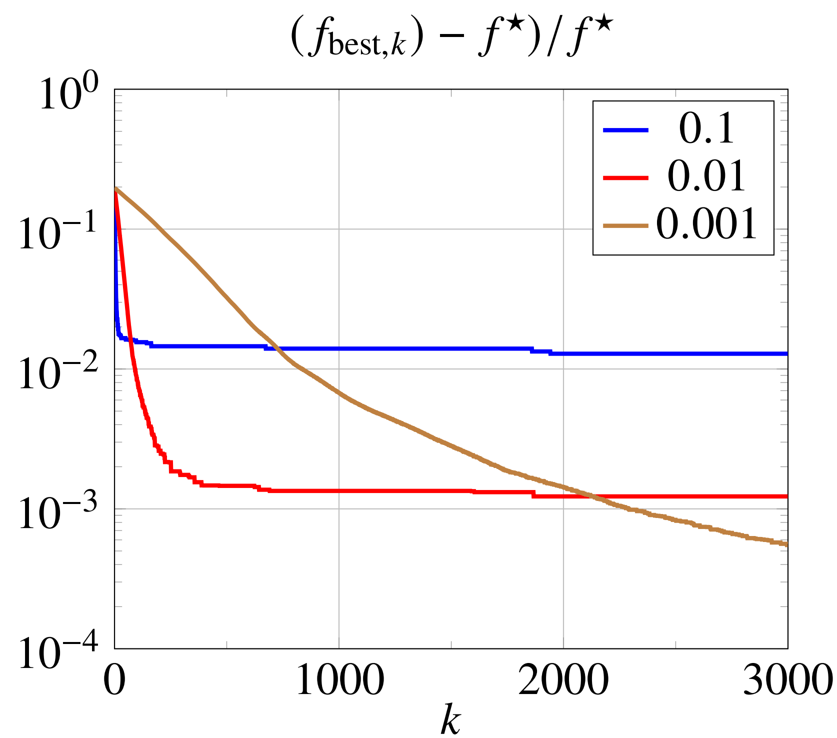
Example: 1-norm minimization

$$\text{minimize } \|Ax - b\|_1$$

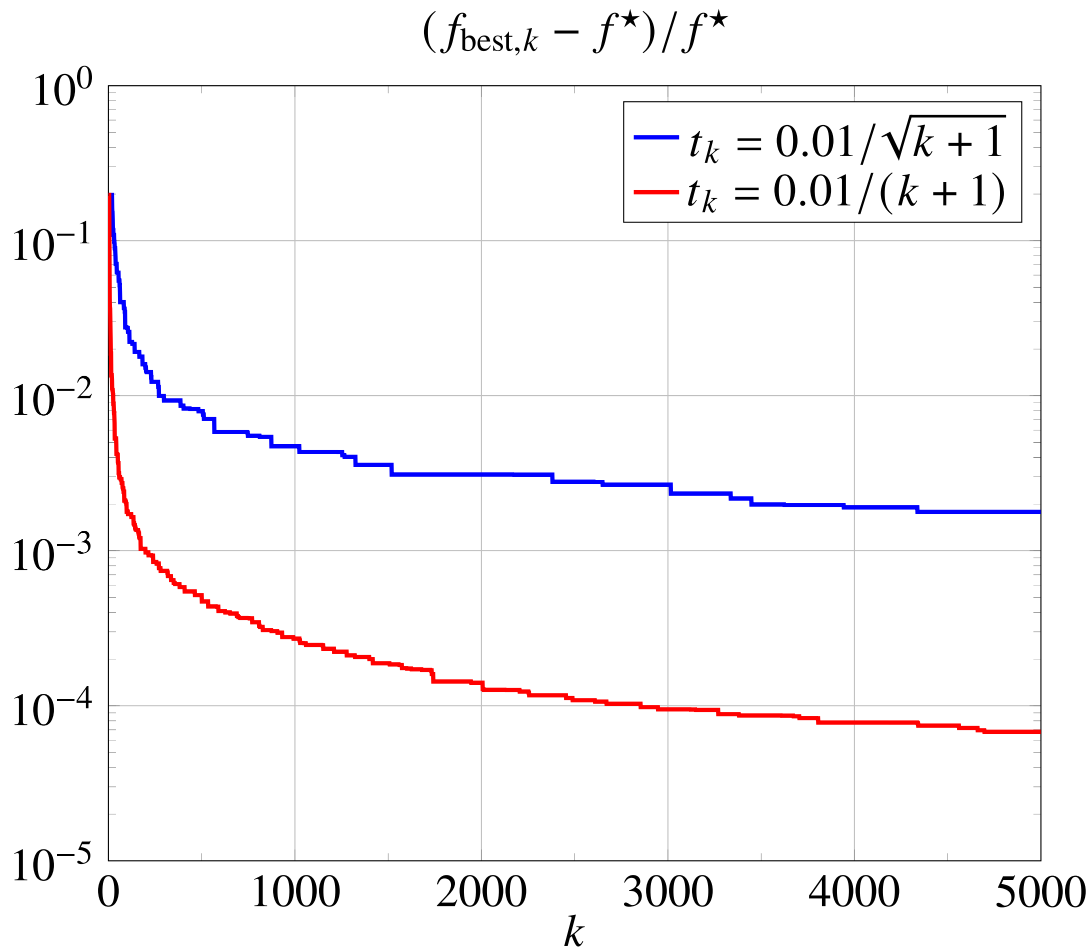
- subgradient is given by $A^T \operatorname{sign}(Ax - b)$
- example with $A \in \mathbf{R}^{500 \times 100}$, $b \in \mathbf{R}^{500}$

Fixed steplength $t_k = s/\|g_k\|_2$ for $s = 0.1, 0.01, 0.001$





Diminishing step size: $t_k = 0.01/\sqrt{k+1}$ and $t_k = 0.01/(k+1)$



Optimal step size for fixed number of iterations

from page 3.5: if $s_i = t_i \|g_i\|_2$ and $\|x_0 - x^\star\|_2 \leq R$, then

$$f_{\mathrm{best},k} - f^\star \leq \frac{R^2 + \sum_{i=0}^{k} s_i^2}{2 \sum_{i=0}^{k} s_i / G}$$

- for given k, the right-hand side is minimized by the fixed step length

$$s_i = s = \frac{R}{\sqrt{k+1}}$$

- the resulting bound after k steps is

$$f_{\mathrm{best},k} - f^\star \leq \frac{GR}{\sqrt{k+1}}$$

- this guarantees an accuracy $f_{\mathrm{best},k} - f^\star \leq \epsilon$ in $k = O(1/\epsilon^2)$ iterations

Optimal step size when $f^\star$ is known

- the right-hand side in the first inequality of page 3.5 is minimized by

$$t_i = \frac{f(x_i) - f^\star}{\|g_i\|_2^2}$$

- the optimized bound is

$$\frac{\left(f(x_i) - f^\star\right)^2}{\|g_i\|_2^2} \leq \|x_i - x^\star\|_2^2 - \|x_{i+1} - x^\star\|_2^2$$

- applying this recursively from $i = 0$ to $i = k$ (and using $\|g_i\|_2 \leq G$) gives

$$f_{\text{best},k} - f^\star \leq \frac{G\|x_0 - x^\star\|_2}{\sqrt{k+1}}$$

Example: find point in intersection of convex sets

find a point in the intersection of m closed convex sets $C_1, \dots, C_m$:

$$\text{minimize} \quad f(x) = \max\{f_1(x), \dots, f_m(x)\}$$

where $f_j(x) = \inf_{y \in C_j} \|x - y\|_2$ is Euclidean distance of x to C_j

- $f^\star = 0$ if the intersection is nonempty
- (from page 2.14) $g \in \partial f(\hat{x})$ if $g \in \partial f_j(\hat{x})$ and C_j is farthest set from $\hat{x}$
- (from page 2.20) subgradient $g \in \partial f_j(\hat{x})$ follows from projection $P_j(\hat{x})$ on C_j:

$$g = 0 \quad \text{if } \hat{x} \in C_j, \qquad g = \frac{1}{\|\hat{x} - P_j(\hat{x})\|_2}\left(\hat{x} - P_j(\hat{x})\right) \quad \text{if } \hat{x} \notin C_j$$

  note that $\|g\|_2 = 1$ if $\hat{x} \notin C_j$

Subgradient method for point in intersection of convex sets

- optimal step size (page 3.11) for $f^\star = 0$ and $\|g_i\|_2 = 1$ is $t_i = f(x_i)$
- at iteration k, find farthest set C_j (with $f(x_k) = f_j(x_k)$), and take

$$
\begin{aligned}
x_{k+1} &= x_k - \frac{f(x_k)}{f_j(x_k)}(x_k - P_j(x_k)) \\
&= P_j(x_k)
\end{aligned}
$$

  at each step, we project the current point onto the farthest set

- a version of the *alternating projections* algorithm
- for $m = 2$, projections alternate onto one set, then the other
- later, we will see faster sequential projection methods that are almost as simple

Projected subgradient method

the subgradient method is easily extended to handle constrained problems

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

where C is a closed convex set

Projected subgradient method: choose $x_0 \in C$ and repeat

$$x_{k+1} = P_C(x_k - t_k g_k), \quad k = 0, 1, \ldots$$

- $P_C(y)$ denotes the Euclidean projection of y on C
- g_k is any subgradient of f at x_k
- t_k is chosen by same step size rules as for unconstrained problem (page 3.2)

Examples of simple convex sets

subgradient projection is practical only if projection on C is easy to compute

Halfspace: $C = \{x \mid a^T x \le b\}$ (with $a \ne 0$)

$$P_C(x) = x + \frac{b - a^T x}{\|a\|_2^2} a \quad \text{if } a^T x > b, \qquad P_C(x) = x \quad \text{if } a^T x \le b$$

Rectangle: $C = \{x \in \mathbf{R}^n \mid l \le x \le u\}$ where $l \le u$

$$P_C(x)_k = \begin{cases} l_k & x_k \le l_k \\ x_k & l_k \le x_k \le u_k \\ u_k & x_k \ge u_k \end{cases}$$

Norm balls: $C = \{x \mid \|x\| \le R\}$ for many common norms (*e.g.*, 236B page 5.26)

we'll encounter many other examples later in the course

Projection on closed convex set

$$P_C(x) = \underset{u \in C}{\operatorname{argmin}} \|u - x\|_2^2$$

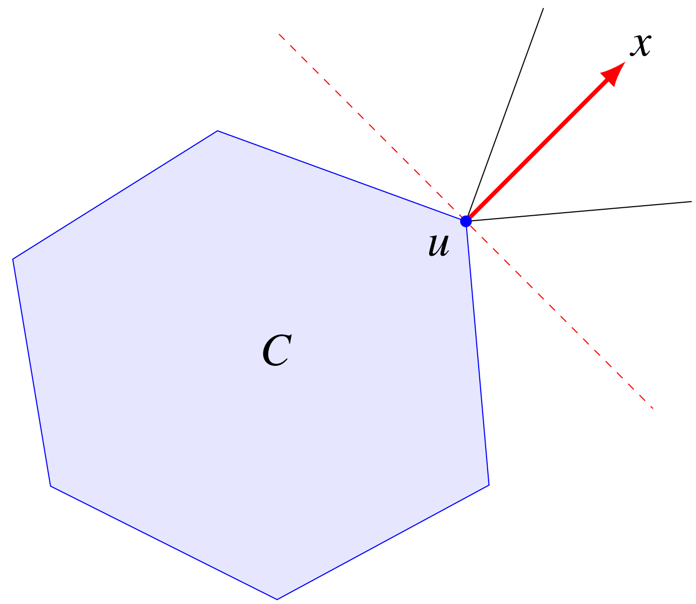
$$u = P_C(x)$$

$$\Updownarrow$$

$$(x - u)^T(z - u) \leq 0 \quad \forall z \in C$$

$$\Updownarrow$$

$$\|x - z\|_2^2 \geq \|x - u\|_2^2 + \|z - u\|_2^2 \quad \forall z \in C$$



this follows from general optimality conditions in 236B page 4.9

Analysis

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

- C is a closed convex set; other assumptions are the same as on page 3.3
- first inequality on page 3.5 still holds:

$$\begin{aligned} \|x_{i+1} - x^\star\|_2^2 &= \left\|P_C(x_i - t_i g_i) - x^\star\right\|_2^2 \\ &\le \left\|x_i - t_i g_i - x^\star\right\|_2^2 \\ &= \|x_i - x^\star\|_2^2 - 2t_i g_i^T(x_i - x^\star) + t_i^2\|g_i\|_2^2 \\ &\le \|x_i - x^\star\|_2^2 - 2t_i\left(f(x_i) - f^\star\right) + t_i^2\|g_i\|_2^2 \end{aligned}$$

  second line follows from page 3.16 (with $z = x^\star$, $x = x_i - t_i g_i$)

- hence, earlier analysis also applies to subgradient projection method

Optimality of the subgradient method

can the $f_{\text{best},k} - f^\star \leq GR/\sqrt{k+1}$ bound on page 3.10 be improved?

Problem class

$$\text{minimize} \quad f(x)$$

- assumptions on page 3.3 are satisfied
- we are given a starting point $x^{(0)}$ with $\|x^{(0)} - x^\star\|_2 \leq R$
- we are given the Lipschitz constant G of f on $\{x \mid \|x - x^\star\|_2 \leq R\}$
- f is defined by an oracle: given x, the oracle returns $f(x)$ and a $g \in \partial f(x)$

Algorithm class

- algorithm can choose any $x^{(i+1)}$ from the set $x^{(0)} + \text{span}\{g^{(0)}, g^{(1)}, \ldots, g^{(i)}\}$
- we stop after a fixed number k of iterations

Test problem and oracle

$$f(x) = \max_{i=1,\ldots,k+1} x_i + \frac{1}{2}\|x\|_2^2 \quad \text{(with } k < n\text{)}, \qquad x^{(0)} = 0$$

- subdifferential $\partial f(x) = \mathrm{conv}\{e_j + x \mid 1 \leq j \leq k+1,\ x_j = \max_{i=1,\ldots,k+1} x_i\}$
- solution and optimal value

$$x^\star = -(\underbrace{\frac{1}{k+1}, \ \ldots, \ \frac{1}{k+1}}_{k+1 \text{ times}}, \ 0, \ldots, 0), \qquad f^\star = -\frac{1}{2(k+1)}$$

- distance of starting point to solution is $R = \|x^{(0)} - x^\star\|_2 = 1/\sqrt{k+1}$
- Lipschitz constant on $\{x \mid \|x - x^\star\|_2 \leq R\}$:

$$G = \sup_{g \in \partial f(x),\ \|x - x^\star\|_2 \leq R} \|g\|_2 \leq \frac{2}{\sqrt{k+1}} + 1$$

- the oracle returns the subgradient $e_{\hat{\jmath}} + x$ where $\hat{\jmath} = \min\{j \mid x_j = \max_{i=1,\ldots,k+1} x_i\}$

Iteration

- after $i \leq k$ iterations of any algorithm in the algorithm class,

$$x^{(i)} = (x_1^{(i)}, \ldots, x_i^{(i)}, 0, \ldots, 0), \qquad f(x^{(i)}) \geq \frac{1}{2}\|x^{(i)}\|_2^2 \geq 0, \qquad f_{\mathrm{best},i} = 0$$

- suboptimality after k iterations

$$f_{\mathrm{best},k} - f^\star = -f^\star = \frac{1}{2(k+1)} = \frac{GR}{2(2+\sqrt{k+1})}$$

Conclusion

- example shows that $O(GR/\sqrt{k})$ bound cannot be improved
- subgradient method is “optimal” (for this problem and algorithm class)

Summary: subgradient method

- handles general nondifferentiable convex problems
- often leads to very simple algorithms
- convergence can be very slow
- no good stopping criterion
- theoretical complexity: $O(1/\epsilon^2)$ iterations to find ϵ-suboptimal point
- an "optimal" first-order method: $O(1/\epsilon^2)$ bound cannot be improved

References

- S. Boyd, *Lecture slides and notes for EE364b, Convex Optimization II*.
- Yu. Nesterov, *Lectures on Convex Optimization* (2018), section 3.2.3. The example on page 3.19 is in §3.2.1.
- B. T. Polyak, *Introduction to Optimization* (1987), section 5.3.