Homework assignment #8

In this problem you are asked to write a MATLAB function for solving ℓ_1-norm approximation problems

$$\text{minimize} \quad \|Pu + q\|_1 \tag{1}$$

via the primal-dual interior-point method of lecture 15. The function will be called as `u = l1(P, q)` and should solve the problem with a relative accuracy of 10^{-6} or an absolute accuracy of 10^{-8}, *i.e.*, the return value u must satisfy

$$\|Pu + q\|_1 - p^\star \leq 10^{-6} \cdot p^\star \qquad \text{or} \qquad \|Pu + q\|_1 - p^\star \leq 10^{-8},$$

where $p^\star$ is the optimal value of (1). You can assume that P is a left-invertible matrix ($\mathbf{rank}\, P = n$ if P is $m \times n$) and that it is dense (a 'full' matrix in MATLAB).

We solve the problem using the primal-dual method of lecture 15 (pages 18–20), applied to the linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T v \\ \text{subject to} & \begin{bmatrix} P & -I \\ -P & -I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} -q \\ q \end{bmatrix}. \end{array} \tag{2}$$

To improve the efficiency we will take advantage of the structure in the constraints.

1. *Initialization.* The primal-dual method can be started at infeasible primal and dual points. However good feasible starting points for the LP (2) are readily available from the solution u_{ls} of the least-squares problem

   $$\text{minimize} \quad \|Pu + q\|$$

   (in MATLAB: `u = -P\q`). As primal starting point we can use $u = u_{\text{ls}}$ and choose v so that we have strict feasibility in (2). To find a strictly feasible point for the dual of (2), we note that the least-squares residual $r_{\text{ls}} = Pu_{\text{ls}} + q$ satisfies $P^T r_{\text{ls}} = 0$. This property can be used to construct a strictly feasible point for the dual of (2).

   If the starting points are strictly feasible, all iterates in the algorithm will remain strictly feasible and we do not have to test the feasibility residuals in the stopping criteria.

2. The most expensive part of each iteration of the primal-dual interior-point method is the solution of two sets of equations of the form

   $$A^T S^{-1} Z A \Delta x = r \tag{3}$$

where S and Z are positive diagonal matrices that change at each iteration (see page 15–14). In our problem,

$$A = \begin{bmatrix} P & -I \\ -P & -I \end{bmatrix}, \qquad \Delta x = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix},$$

so (3) has $m+n$ equations in $m+n$ variables if P has size $m \times n$. By exploiting the structure of A, show that systems of the form (3) can be reduced to a smaller system

$$P^T D P \Delta u = \tilde{r}$$

with D diagonal, followed by a number of inexpensive operations.

3. Test your code on randomly generated problem instances. To generate P and q, you can use the commands

```
P = randn(m, n);  q = randn(m, 1);.
```

Submit your solution (as a file `l1.m`) by email to Yifan at `ysun01@ucla.edu` with subject line 'EE236A homework 8 submission'. You do not need to submit a paper copy. We will test the code on problems of dimension up to $m = 2000$, $n = 1000$.