Lecture 4
Convexity

- convex hull
- polyhedral cone
- decomposition

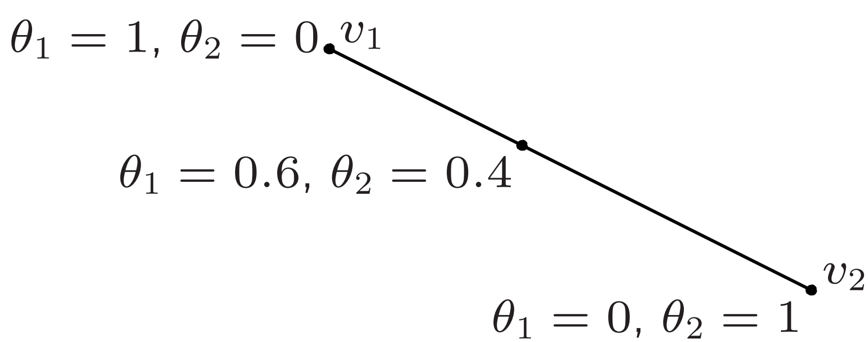
Convex combination

a **convex combination** of points $v_1, \ldots, v_k$ is a linear combination

$$x = \theta_1 v_1 + \theta_2 v_2 + \cdots + \theta_k v_k$$

with $\theta_i \geq 0$ and $\sum_{i=1}^{k} \theta_i = 1$

for $k = 2$, the point x is in the **line segment** with endpoints v_1, v_2



Convex set

a set S is **convex** if it contains all convex combinations of points in S

examples

- affine sets: if $Cx = d$ and $Cy = d$, then

$$C(\theta x + (1-\theta)y) = \theta Cx + (1-\theta)Cy = d \qquad \forall \theta \in \mathbf{R}$$

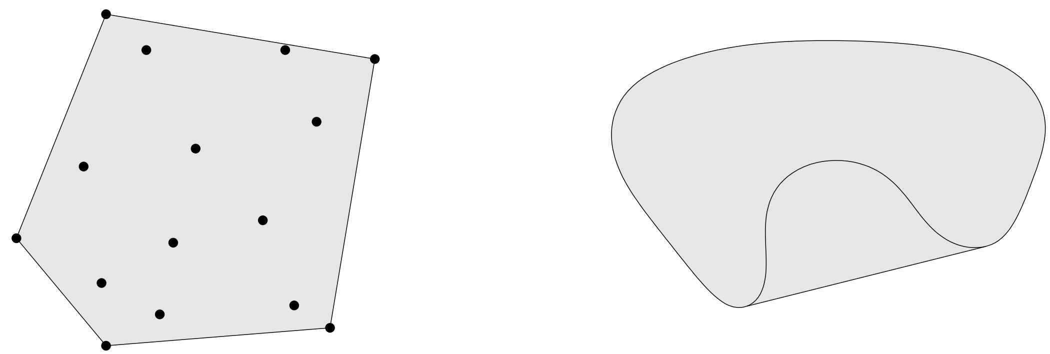
- polyhedra: if $Ax \le b$ and $Ay \le b$, then

$$A(\theta x + (1-\theta)y) = \theta Ax + (1-\theta)Ay \le b \qquad \forall \theta \in [0,1]$$

Convex hull and polytope

convex hull of a set S: the set of all convex combinations of points in S

notation: $\operatorname{conv} S$



polytope: the convex hull $\operatorname{conv}\{v_1, v_2, \ldots, v_k\}$ of a finite set of points

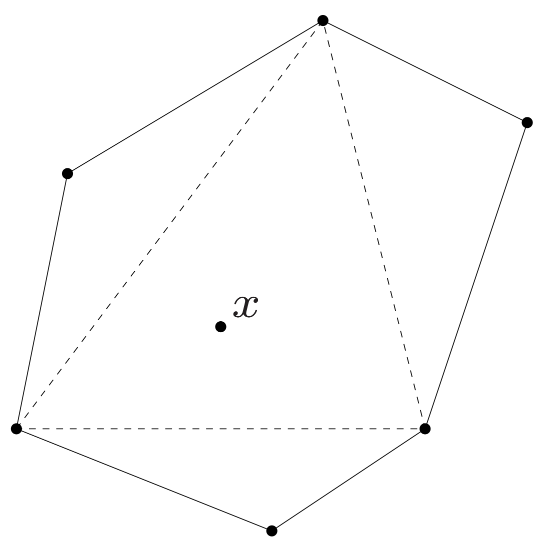
(the first set in the figure is an example)

Exercise: Carathéodory's theorem

by definition, $\mathrm{conv}(S)$ is the set of points x that can be expressed as

$$x = \theta_1 v_1 + \cdots + \theta_k v_k \quad \text{with} \quad \sum_{i=1}^{k} \theta_i = 1, \quad \theta_i \geq 0, \quad v_1, \ldots, v_k \in S$$

show that if $S \subseteq \mathbf{R}^n$ then k can be taken less than or equal to $n+1$



in $\mathbf{R}^2$, every $x \in \mathrm{conv}\, S$ can be written as a convex combination of 3 points in S

solution: start from any convex decomposition of x:

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix}, \qquad \theta_i \geq 0, \quad i = 1, \ldots, m$$

let P be the set of vectors $\theta = (\theta_1, \ldots, \theta_m)$ that satisfy these conditions

- P is a nonempty polyhedron, described in 'standard form' (page 3–27)
- if $\hat{\theta} \in P$ is an extreme point of P, then (from page 3–27)

$$\text{rank}(\begin{bmatrix} v_{i_1} & v_{i_2} & \cdots & v_{i_k} \\ 1 & 1 & \cdots & 1 \end{bmatrix} = k$$

  where $\{i_1, \ldots, i_k\} = \{i \mid \hat{\theta}_i > 0\}$

- the rank condition implies $k \leq n + 1$

Convex cone

convex cone: a nonempty set S with the property

$$x_1, \ldots, x_k \in S, \quad \theta_1 \geq 0, \ldots, \theta_k \geq 0 \qquad \Longrightarrow \qquad \theta_1 x_1 + \cdots + \theta_k \in S$$

- all **nonnegative combinations** of points in S are in S
- S is a convex set and a cone (*i.e.*, $\alpha x \in S$ implies $\alpha x \in S$ for $\alpha \geq 0$)

examples

- subspaces
- a polyhedral cone: a set defined as

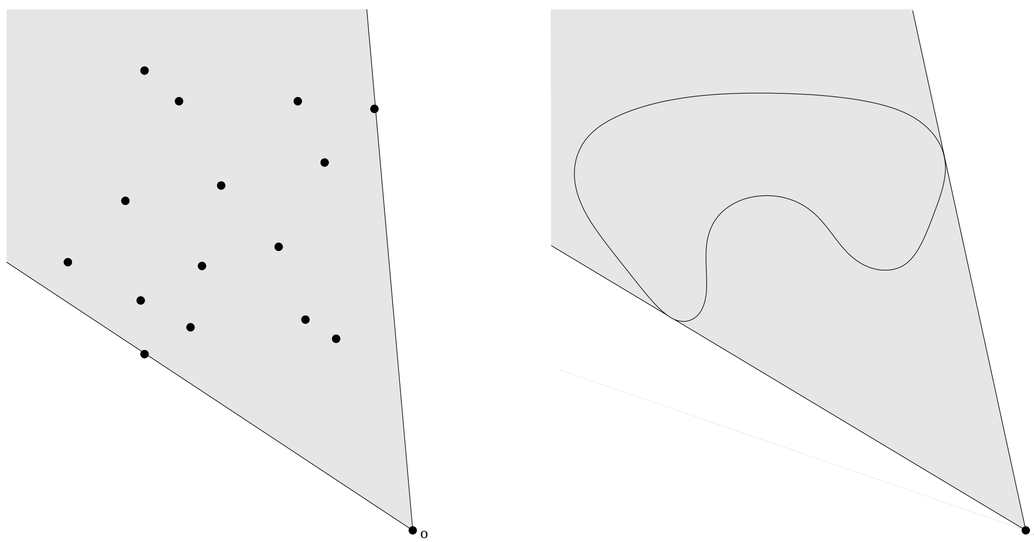
$$S = \{x \mid Ax \leq 0, \; Cx = 0\}$$

(the solution of a finite system of homogeneous linear inequalities)

Conic hull

conic hull of a set S: set of all nonnegative combinations of points in S

- also known as the *cone generated by* S
- notation: $\mathbf{cone}\, S$

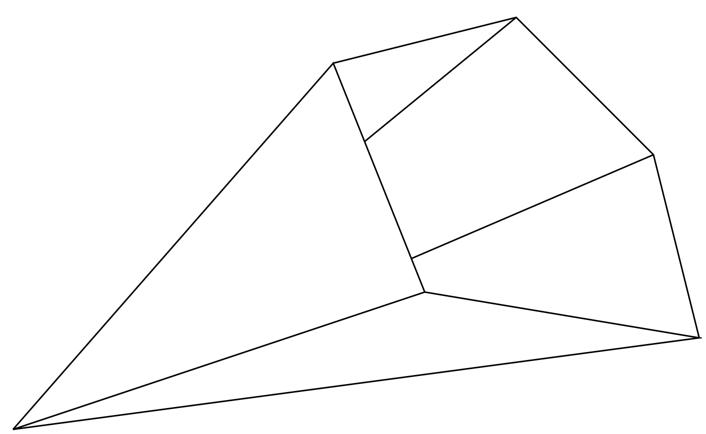


finitely generated cone: the conic hull $\mathbf{cone}\{v_1, v_2, \ldots, v_k\}$ of a finite set

Pointed polyhedral cone

consider a polyhedral cone $K = \{x \in \mathbf{R}^n \mid Ax \leq 0,\ Cx = 0\}$

- the lineality space is the nullspace of $\begin{bmatrix} A \\ C \end{bmatrix}$
- K is pointed if $\begin{bmatrix} A \\ C \end{bmatrix}$ has rank n
- if K is pointed, it has one extreme point (the origin)
- the one-dimensional faces are called **extreme rays**



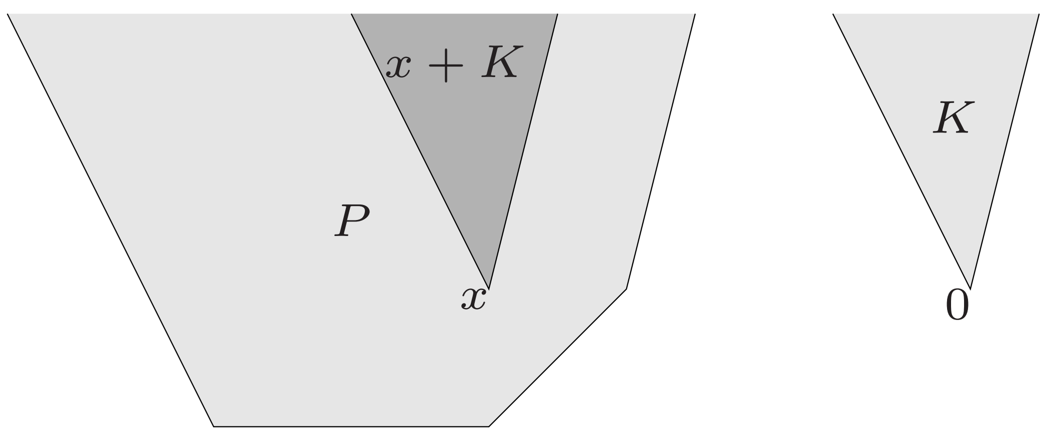
Recession cone

the **recession cone** of a polyhedron $P = \{x \mid Ax \leq b,\, Cx = d\}$ is

$$K = \{y \mid Ay \leq 0,\, Cy = 0\}$$

(also known as the **asymptotic cone** of P)

- K has the same lineality space as P
- K is pointed if and only if P is pointed
- if $x \in P$ then $x + y \in P$ for all $y \in K$



Decomposition

every polyhedron P can be decomposed as

$$P = L + Q = L + \mathrm{conv}\{v_1, \ldots, v_r\} + \mathrm{cone}\{w_1, \ldots, w_s\}$$

- L is the lineality space
- Q is a pointed polyhedron
- $v_1, \ldots, v_r$ are the extreme points of Q
- $w_1, \ldots, w_s$ generate the extreme rays of the recession cone of Q

(we'll skip the proof)

applications

- useful for theoretical purposes
- in general, extremely costly to compute from inequality description of P
- implicitly used by some algorithms