Lecture 8
Linear-fractional optimization

• linear-fractional program

• generalized linear-fractional program

• examples
Linear-fractional program

minimize \( \frac{c^T x + d}{g^T x + h} \)
subject to \( Ax \leq b \)
\( g^T x + h \geq 0 \)

• if needed, we interpret \( a/0 \) as \( a/0 = +\infty \) if \( a > 0 \), \( a/0 = -\infty \) if \( a \leq 0 \)

• however, in most applications, \( Ax \leq b \) implies \( g^T x + h > 0 \)

**equivalent form** (with added variable \( \alpha \))

minimize \( \alpha \)
subject to \( c^T x + d \leq \alpha(f^T x + g) \)
\( Ax \leq b \)
\( f^T x + g \geq 0 \)
Level sets

\[ C_\alpha = \{ x \mid g^T x + h > 0, \frac{c^T x + d}{g^T x + h} = \alpha \} \]
\[ = \{ x \mid g^T x + h > 0, (c - \alpha g)^T x = \alpha h - d \} \]

five level sets with

\[ \alpha_1 > 0 > \alpha_2 > \alpha_3 > \alpha_4 \]
Geometrical interpretation

Optimal value $\alpha^*$ attained at $x^*$

Finite optimal value $\alpha^*$, not attained
Equivalent linear program

LFP: minimize \[ \frac{c^T x + d}{g^T x + h} \]
subject to \[ Ax \leq b \]
\[ g^T x + h \geq 0 \]

LP: minimize \[ c^T y + td \]
subject to \[ Ay \leq tb \]
\[ g^T y + th = 1 \]
\[ t \geq 0 \]

we will assume that \( g^T x + h > 0 \) for all \( x \in P = \{ x \mid Ax \leq b \} \)

- nonlinear change of variables maps \( x \in P \) to feasible \((y, t)\) with \( t > 0 \):
  \[ y = \frac{1}{g^T x + h} x, \quad t = \frac{1}{g^T x + h} \]

- inverse transformation \( x = y/t \) maps feasible \((y, t)\) with \( t > 0 \) to \( x \in P \)

- change of variables and its inverse preserve objective values:
  \[ \frac{(c^T x + d)}{(g^T x + h)} = c^T y + td \]
Interpretation of $t = 0$

suppose $(y, t)$ is feasible for the LP with $t = 0$ (i.e., $Ay \leq 0$, $g^T y = 1$)

- $(y, t)$ does not correspond to a point $x \in P$ ($x = y/t$ is not defined)
- $y$ can be interpreted as the direction of a half-line based at any $\hat{x} \in P$
  \[
  \{ \hat{x} + \lambda y \mid \lambda \geq 0 \}
  \]
- this half-line is in $P$:
  \[
  A(\hat{x} + \lambda y) \leq b, \quad g^T(\hat{x} + \lambda y) + h \geq 0 \quad \text{for all } \lambda \geq 0
  \]
- the LFP objective approaches the LP objective $c^T y$ asymptotically:
  \[
  \lim_{\lambda \to \infty} \frac{c^T(\hat{x} + \lambda y) + d}{g^T(\hat{x} + \lambda y) + h} = c^T y
  \]
Generalized linear-fractional programming

minimize \( \max_{i=1,\ldots,m} \frac{c_i^T x + d_i}{f_i^T x + g_i} \)

subject to \( Ax \leq b \)
\( f_i^T x + g_i \geq 0, \quad i = 1, \ldots, m \)

**equivalent formulation** (with auxiliary variable \( \alpha \in \mathbb{R} \))

minimize \( \alpha \)

subject to \( Cx + d \leq \alpha (Fx + g) \)
\( Ax \leq b \)
\( Fx + g \geq 0 \)

- \( C \) and \( F \) are matrices with rows \( c_i^T, f_i^T \)
- in contrast to LFP of p. 8–2, generalized LFP is not reducible to an LP
- can be solved efficiently as a sequence of LP feasibility problems
Sublevel sets

**definition:** $\alpha$-sublevel set of objective function is

$$S_\alpha = \{x \mid \max_{i=1,...,m} \frac{c_i^T x + d_i}{f_i^T x + g_i} \leq \alpha, \ F x + g \geq 0\}$$

$$= \{x \mid C x + d \leq \alpha(F x + g), \ F x + g \geq 0\}$$

(with $a/0$ interpreted as on page 8–2)

**properties**

- $S_\alpha$ is a polyhedron
- the sublevel sets $S_\alpha$ are nested: if $\alpha < \beta$ then $S_\alpha \subseteq S_\beta$:

$$\begin{align*}
Cx + d &\leq \alpha(F x + g) \\
F x + g &\geq 0
\end{align*} \implies
Cx + d \leq \beta(F x + g)$$
Bisection algorithm

algorithm
given: interval \([l, u]\) of width \(\epsilon_0 = u - l\) that contains the optimal \(\alpha\)
repeat until \(u - l \leq \epsilon\):
  • take \(\alpha = (u + l)/2\) and solve the feasibility problem
    
    \[
    \begin{align*}
    \text{find} & \quad x \\
    \text{subject to} & \quad Cx + d \leq \alpha(Fx + g) \\
    & \quad Ax \leq b \\
    & \quad Fx + g \geq 0
    \end{align*}
    \]
    
    • if feasible, take \(u := \alpha\); if infeasible, take \(l := \alpha\)

convergence

• after each update, interval \([l, u]\) contains optimal \(\alpha\)
• width \(u - l\) is halved at each step, so \#iterations = \(\left\lceil \log_2(\epsilon_0/\epsilon) \right\rceil\)
Von Neumann economic growth problem

• simple model of an economy with \( m \) commodities, \( n \) activities (sectors)

• \( x_i(t) \) is ‘intensity’ of activity \( i \) in period \( t \)

• \( a_i^T x(t) \): amount of commodity \( i \) consumed in period \( t \)

• \( b_i^T x(t) \): amount of commodity \( i \) produced in period \( t \)

maximize growth rate of economy (variables \( x(t) \), \( x(t + 1) \)):

\[
\text{maximize} \quad \min_{i=1,...,n} \frac{x_i(t + 1)}{x_i(t)}
\]

subject to

\[
\begin{align*}
Ax(t + 1) & \leq Bx(t) \\
x(t) & \geq 1
\end{align*}
\]

• cost function is growth rate of sector with slowest growth rate

• a generalized linear-fractional problem
Optimal transmitter power allocation

- $m$ transmitters, $mn$ receivers all at same frequency
- $n$ receivers labeled $(i, j), j = 1, \ldots, n$, listen to transmitter $i$
- transmitters $k \neq i$ interfere at receivers $(i, j)$

**variables:** transmit powers $p_i$

**objective:** maximize worst signal to noise-plus-interference ratio
signal to noise-plus-interference ratio at receiver \((i, j)\): 

\[
\text{SINR}_{ij}(p) = \frac{A_{ij} p_i}{\sum_{k \neq i} A_{ijk} p_k + N_{ij}}
\]

- \(A_{ijk}\) is path gain from transmitter \(k\) to receiver \((i, j)\)
- \(N_{ij}\) is (self) noise power of receiver \((i, j)\)

**optimization problem**

\[
\begin{align*}
\text{maximize} \quad & \min_{ij} \text{SINR}_{ij}(p) \\
\text{subject to} \quad & 0 \leq p_i \leq p_{\text{max}}, \quad i = 1, \ldots, m
\end{align*}
\]

a (generalized) linear-fractional optimization problem in the variables \(p\)