Lecture 17
Network flow optimization

• minimum cost network flows

• total unimodularity

• examples
Networks

network (directed graph, digraph): $m$ nodes connected by $n$ directed arcs

- arcs are ordered pairs $(i, j)$ of nodes
- we assume there is at most one arc from node $i$ to node $j$
- there are no loops (arcs $(i, i)$)

arc-node incidence matrix: $m \times n$ matrix $A$ with entries

$$A_{i,j} = \begin{cases} 
1 & \text{if arc } j \text{ starts at node } i \\
-1 & \text{if arc } j \text{ ends at node } i \\
0 & \text{otherwise}
\end{cases}$$
Example

\[ A = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\end{bmatrix} \]
Network flow

flow vector \( x \in \mathbb{R}^n \)

- \( x_j \): flow (of material, traffic, charge, information, . . .) through arc \( j \)
- positive if in direction of arc; negative otherwise

total flow leaving node \( i \):

\[
\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i
\]
External supply

**supply vector** $b \in \mathbb{R}^m$

- $b_i$ is external supply at node $i$ (negative $b_i$ represents external demand)
- must satisfy $1^T b = 0$ (total supply = total demand)

![Diagram of network flow with nodes and arrows showing flow from node $i$ to other nodes]

**balance equations:**

$$Ax = b$$
Minimum cost network flow problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad l \leq x \leq u
\end{align*}
\]

- \(c_i\) is unit cost of flow through arc \(i\)
- \(l_j\) and \(u_j\) are limits on flow through arc \(j\) (typically, \(l_j \leq 0, u_j \geq 0\))
- we assume \(l_j < u_j\), but allow \(l_j = -\infty\) and \(u_j = \infty\) to simplify notation

includes many network optimization problems as special cases
Maximum flow problem

maximize flow from node 1 (source) to node $m$ (sink) through the network

maximize $t$ subject to $Ax = te$

$l \leq x \leq u$

where $e = (1, 0, \ldots, 0, -1)$
Formulation as minimum cost flow problem

minimize \(-t\)
subject to 
\[
\begin{bmatrix}
A & -e
\end{bmatrix}
\begin{bmatrix}
x \\
t
\end{bmatrix} = 0
\]
\[l \leq x \leq u\]
Outline

- minimum cost network flows
- total unimodularity
- examples
Totally unimodular matrix

A matrix is **totally unimodular** if all its minors are $-1$, $0$, or $1$  
(a minor is the determinant of a square submatrix)

**Examples**

- the matrix

\[
\begin{bmatrix}
1 & 0 & -1 & 0 & 1 \\
0 & -1 & 1 & -1 & -1 \\
0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

- node-arc incidence matrix of a directed graph (proof on next page)

**Properties** of a totally unimodular matrix $A$

- the entries $A_{ij}$ (i.e., its minors of order 1) are $-1$, $0$, or $1$
- the inverse of any nonsingular square submatrix of $A$ has entries $\pm 1$, $0$
proof: let $A$ be an $m \times n$ node-arc incidence matrix

- the entries of $A$ are $-1$, $0$, or $1$
- $A$ has exactly two nonzero entries ($-1$ and $1$) per column

consider a $k \times k$ submatrix $B$ of $A$

- if $B$ has a zero column, its determinant is zero
- if all columns of $B$ have two nonzero entries, then $1^T B = 0$, $\det B = 0$
- otherwise $B$ has a column, say column $j$, with one nonzero entry $B_{ij}$, so

$$\det B = (-1)^{i+j} B_{ij} \det C$$

$C$ is square of order $k - 1$, obtained by deleting row $i$ and column $j$ of $B$

hence, can show by induction on $k$ that all minors of $A$ are $\pm 1$ or $0$
Integrality of extreme points

let $P$ be a polyhedron in $\mathbb{R}^n$ defined by

$$Ax = b, \quad l \leq x \leq u$$

where

- $A$ is totally unimodular
- $b$ is an integer vector
- the finite lower bounds $l_k$ and finite upper bounds $u_k$ are integers

then all the extreme points of $P$ are integer vectors
**proof:** apply rank test to determine whether \( \hat{x} \in P \) is an extreme point

- partition \( \{1, 2, \ldots, n\} \) in three sets \( J_0, J_-, J_+ \) with
  
  \[ l_k < \hat{x}_k < u_k \quad \text{for} \quad k \in J_0 \]
  \[ \hat{x}_k = l_k \quad \text{for} \quad k \in J_- \]
  \[ \hat{x}_k = u_k \quad \text{for} \quad k \in J_+ \]

  let \( A_0, A_-, A_+ \) be the submatrices of \( A \) with columns in \( J_0, J_-, J_+ \)

- \( \hat{x} \) is an extreme point if and only if

  \[
  \text{rank} \begin{bmatrix}
  0 & I & 0 \\
  0 & 0 & -I \\
  A_0 & A_- & A_+
  \end{bmatrix} = n \iff A_0 \text{ has full column rank}
  \]

  integrality of \( \hat{x} \) then follows from \( A_0\hat{x}_{J_0} = b - A_-\hat{x}_{J_-} - A_+\hat{x}_{J_+} \)

- right-hand side is an integral vector (\( \hat{x}_k \) is integer for \( k \in J_- \cup J_+ \))

- inverse of any nonsingular submatrix of \( A_0 \) has integer entries
Implications for combinatorial optimization

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad l \leq x \leq u \\
& \quad x \in \mathbb{Z}^n
\end{align*}
\]

• an integer linear program, very difficult in general
• equivalent to its linear program relaxation

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad l \leq x \leq u
\end{align*}
\]

if \(A\) is totally unimodular and \(b, l, u\) are integer vectors

(extreme optimal solution of the relaxation is optimal for the integer LP)
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Shortest path problem

shortest path in directed graph with node-arc incidence matrix $A$

- (forward) paths from node 1 to $m$ can be represented by vectors $x$ with

$$Ax = (1, 0, \ldots, 0, -1), \quad x \in \{0, 1\}^n$$

- shortest path is solution of

$\begin{align*}
\text{minimize} & \quad \mathbf{1}^T x \\
\text{subject to} & \quad Ax = (1, 0, \ldots, 0, -1) \\
& \quad x \in \{0, 1\}^n
\end{align*}$

LP formulation

$\begin{align*}
\text{minimize} & \quad \mathbf{1}^T x \\
\text{subject to} & \quad Ax = (1, 0, \ldots, 0, -1) \\
& \quad 0 \leq x \leq 1
\end{align*}$

extreme optimal solutions satisfy $x_i \in \{0, 1\}$
Birkhoff theorem

doubly stochastic matrix: \(N \times N\) matrices \(X\) with \(0 \leq X_{ij} \leq 1\) and

\[
\sum_{i=1}^{N} X_{ij} = 1, \quad j = 1, \ldots, N, \quad \sum_{j=1}^{N} X_{ij} = 1, \quad i = 1, \ldots, N
\]

set of doubly stochastic matrices is a polyhedron \(P\) in \(\mathbb{R}^{N \times N}\)

**Theorem** (p.3–29): the extreme points of \(P\) are the permutation matrices

proof: interpret \(X\) as network flow

- \(N\) input nodes, \(N\) output nodes
- \(X_{ij}\) is flow from input \(i\) to output \(j\)

hence extreme \(X\) has integer entries

example \((N = 3)\)
Weighted bipartite matching

• match $N$ persons to $N$ tasks
• each person assigned to one task; each task assigned to one person
• cost of matching person $i$ to task $j$ is $A_{ij}$

LP formulation

\[
\begin{align*}
\text{minimize} & \quad \sum_{i,j=1}^{N} A_{ij}X_{ij} \\
\text{subject to} & \quad \sum_{i=1}^{N} X_{ij} = 1, \quad j = 1, \ldots, N \\
& \quad \sum_{j=1}^{N} X_{ij} = 1, \quad i = 1, \ldots, N \\
& \quad 0 \leq X_{ij} \leq 1, \quad i, j = 1, \ldots, N
\end{align*}
\]

integrality: extreme optimal solution $X$ has entries $X_{ij} \in \{0, 1\}$