Lecture 12
Simplex method

- adjacent extreme points
- one simplex iteration
- cycling
- initialization
- implementation

Problem format and assumptions

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

A has size $m \times n$

assumption: the feasible set is nonempty and **pointed** ($\mathrm{rank}(A) = n$)

- sufficient condition: for each x_k, the constraints include simple bounds

$$x_k \geq l_k \qquad \text{and/or} \qquad x_k \leq u_k$$

- if needed, can replace 'free' variable x_k by two nonnegative variables

$$x_k = x_k^+ - x_k^-, \qquad x_k^+ \geq 0, \qquad x_k^- \geq 0$$

Simplex method

- invented in 1947 (George Dantzig)
- usually developed for LPs in standard form ('primal' simplex method)
- we will outline the 'dual' simplex method (for inequality form LP)

one iteration:

move from an extreme point to an adjacent extreme point with lower cost

questions

1. how are extreme points characterized? (see lecture 3)
2. how do we find an adjacent extreme point with lower cost?
3. when does the iteration terminate?
4. how do we find an initial extreme point?

Extreme points

recall rank test: to check whether $\hat{x}$ is an extreme point of solution set of

$$a_i^T x \leq b_i, \quad i = 1, \ldots, m$$

- check that $\hat{x}$ satisfies the inequalities
- find the active constraints at $\hat{x}$,

$$J = \{i_1, \ldots, i_k\} = \{i \mid a_i^T \hat{x} = b_i\},$$

  and define the submatrix

$$A_J = \begin{bmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_k}^T \end{bmatrix}$$

- $\hat{x}$ is an extreme point if and only if $\operatorname{rank}(A_J) = n$

Degeneracy

extreme point x is **nondegenerate** if exactly n inequalities are active at it

- A_J is square ($|J| = n$) and nonsingular
- therefore x can be written as $x = A_J^{-1} b_J$, where $b_J = (b_{i_1}, b_{i_2}, \ldots, b_{i_n})$

an extreme point is **degenerate** if more than n inequalities are active at x

note:

- extremality is a *geometric* property (of the set $\mathcal{P} = \{x \mid Ax \leq b\}$)
- (non-)degeneracy also depends on the *description* of $\mathcal{P}$ ($i.e.$, A and b)

until p. 12–20 we assume that all extreme points of $\mathcal{P}$ are nondegenerate

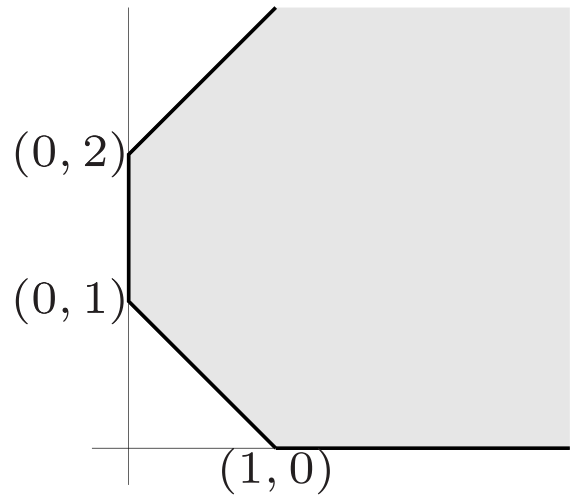
Adjacent extreme points

definition:

extreme points are **adjacent** if they have $n-1$ common active constraints

example

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$



x	$b - Ax$	J
$(1,0)$	$(0,0,1,3)$	$\{1,2\}$
$(0,1)$	$(1,0,0,1)$	$\{2,3\}$
$(0,2)$	$(2,1,0,0)$	$\{3,4\}$

Moving to an adjacent extreme point

given: extreme point x with active index set J, and an index $k \in J$

problem: find adjacent extreme point $\hat{x}$ with active set containing $J \setminus \{k\}$

1. solve the set of n equations in n variables

$$a_i^T \Delta x = 0, \quad i \in J \setminus \{k\}, \qquad a_k^T \Delta x = -1$$

2. if $A\Delta x \leq 0$, then $\{x + \alpha \Delta x \mid \alpha \geq 0\}$ is a feasible half-line:

$$A(x + \alpha \Delta x) \leq b \quad \forall \alpha \geq 0$$

3. else, take $\hat{x} = x + \hat{\alpha}\Delta x$ where $\hat{\alpha} = \max\{\alpha \mid A(x + \alpha \Delta x) \leq b\}$, *i.e.*,

$$\hat{\alpha} = \min_{i : a_i^T \Delta x > 0} \frac{b_i - a_i^T x}{a_i^T \Delta x}$$

discussion

- equations in step 1 are solvable because A_J is nonsingular
- $\hat{\alpha}$ computed in step 3 is positive: $A_J \Delta x \leq 0$ by construction, so

$$a_i^T \Delta x > 0 \quad \Longrightarrow \quad i \notin J \quad \Longrightarrow \quad a_i^T x < b_i$$

- $\hat{x} = x + \hat{\alpha}\Delta x$ is feasible with active constraints $\widehat{J} = (J \setminus \{k\}) \cup I$, where

$$I = \{i \mid a_i^T \Delta x > 0, \; \frac{b_i - a_i^T x}{a_i^T \Delta x} = \hat{\alpha}\}$$

- $\hat{x}$ is an extreme point ($\mathrm{rank}(A_{\widehat{J}}) = n$): take any $j \in I$; since

$$a_j^T \Delta x > 0, \qquad a_i^T \Delta x = 0 \quad \text{for } i \in J \setminus \{k\}$$

  the vector a_j is linearly independent of the vectors a_i, $i \in J \setminus \{k\}$
- by nondegeneracy assumption, $|I| = 1$ (minimizer in step 3 is unique)

Example

find the extreme points adjacent to $x = (1, 0)$ (for example on p. 12–6)

1. try to remove $k = 1$ from active set $J = \{1, 2\}$

   - compute Δx

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \Delta x = (-1, 1)$$

   - minimum ratio test: $A\Delta x = (-1, 0, 1, 2)$

$$\hat{\alpha} = \min\{\frac{b_3 - a_3^T x}{a_3^T \Delta x}, \frac{b_4 - a_4^T x}{a_4^T \Delta x}\} = \min\{\frac{1}{1}, \frac{3}{2}\} = 1$$

   new extreme point: $\hat{x} = (0, 1)$ with active set $\widehat{J} = \{2, 3\}$

2. try to remove $k = 2$ from active set $J = \{1, 2\}$

   - compute Δx

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (1, 0)$$

   - $A\Delta x = (0, -1, -1, -1)$:

$$\{(1, 0) + \alpha(1, 0) \mid \alpha \geq 0\}$$

     is an unbounded edge of the feasible set

Finding an adjacent extreme point with lower cost

given extreme point x with active constraint set J

1. define $z \in \mathbf{R}^m$ with

$$A_J^T z_J + c = 0, \qquad z_j = 0 \quad \text{for } j \notin J$$

2. if $z \geq 0$, then x, z are primal and dual optimal

3. otherwise select k with $z_k < 0$ and determine Δx as on page 12–6:

$$\begin{aligned} c^T(x + \alpha \Delta x) &= c^T x - \alpha z_J^T A_J \Delta x \\ &= c^T x + \alpha z_k \end{aligned}$$

   cost decreases in the direction Δx

One iteration of the simplex method

given an extreme point x with active set J

1. compute $z \in \mathbf{R}^m$ with

$$A_J^T z_J + c = 0, \qquad z_j = 0 \quad \text{for } j \notin J$$

   if $z \geq 0$, terminate: x, z are primal, dual optimal

2. choose k with $z_k < 0$ and compute $\Delta x \in \mathbf{R}^n$ with

$$a_i^T \Delta x = 0 \quad \text{for } i \in J \setminus \{k\}, \qquad a_k^T \Delta x = -1$$

   if $A\Delta x \leq 0$, terminate: LP is unbounded ($p^\star = -\infty$)

3. set $J := J \setminus \{k\} \cup \{j\}$ and $x := x + \hat{\alpha}\Delta x$ where

$$j = \operatorname*{argmin}_{i:a_i^T \Delta x > 0} \frac{b_i - a_i^T x}{a_i^T \Delta x}, \qquad \hat{\alpha} = \frac{b_j - a_j^T x}{a_j^T \Delta x}$$

Pivot selection and convergence

step 2: which k do we choose if z_k has several negative components?

many variants:

- choose most negative z_k
- choose maximum decrease in cost αz_k
- choose smallest k

all three variants work (if extreme points are nondegenerate)

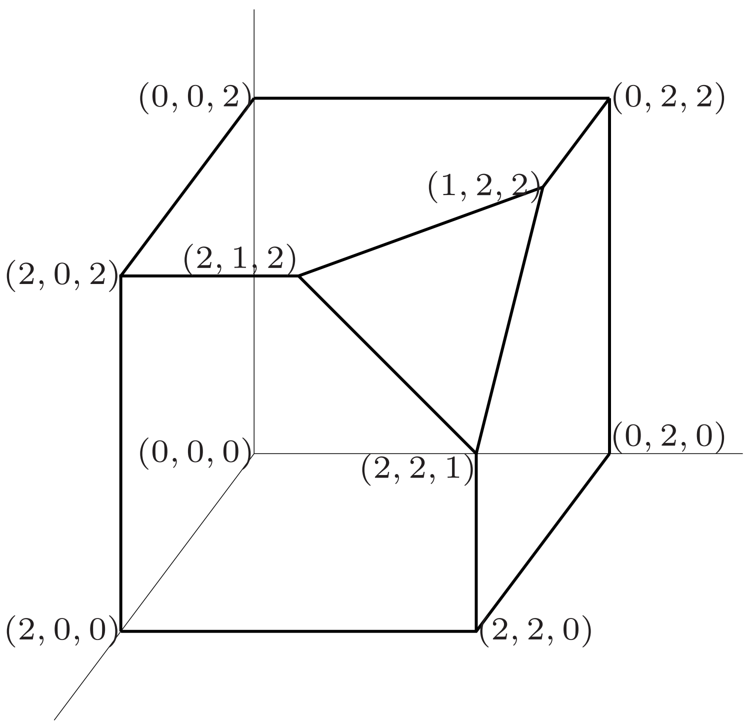
step 3: j is unique and $\hat{\alpha} > 0$ (if all extreme points are nondegenerate)

convergence follows from:

- finiteness of number of extreme points
- strict decrease in cost function at each step

Example

$$\text{min. } x_1 + x_2 - x_3 \qquad \text{s.t.} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 2 \\ 5 \end{bmatrix}$$



- optimal point is $x = (0, 0, 2)$
- start simplex method at $x = (2, 2, 0)$

iteration 1: $x = (2,2,0), \quad b - Ax = (2,2,0,0,0,2,1), \quad J = \{3,4,5\}$

1. compute z:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \\ z_5 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0,0,-1,-1,-1,0,0)$$

   not optimal; remove $k = 3$ from active set

2. compute Δx

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (0,0,1)$$

3. minimum ratio test: $A\Delta x = (0,0,-1,0,0,1,1)$

$$\hat{\alpha} = \operatorname{argmin}\{2/1, 1/1\} = 1, \qquad j = 7$$

iteration 2: $x = (2, 2, 1)$, $\quad b - Ax = (2, 2, 1, 0, 0, 1, 0)$, $\quad J = \{4, 5, 7\}$

1. compute z:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 \\ z_7 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 0, 0, -2, -2, 0, 1)$$

   not optimal; remove $k = 5$ from active set

2. compute Δx

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (0, -1, 1)$$

3. minimum ratio test: $A\Delta x = (0, 1, -1, 0, -1, 1, 0)$

$$\hat{\alpha} = \mathrm{argmin}\{2/1, 1/1\} = 1, \qquad j = 6$$

iteration 3: $x = (2, 1, 2)$, $\quad b - Ax = (2, 1, 2, 0, 1, 0, 0)$, $\quad J = \{4, 6, 7\}$

1. compute z:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_4 \\ z_6 \\ z_7 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 0, 0, 0, 0, 2, -1)$$

   not optimal; remove $k = 7$ from active set

2. compute Δx

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (0, -1, 0)$$

3. minimum ratio test: $A\Delta x = (0, 1, 0, 0, -1, 0, -1)$

$$\hat{\alpha} = \operatorname{argmin}\{1/1\} = 1, \qquad j = 2$$

iteration 4: $x = (2, 0, 2)$, $\quad b - Ax = (2, 0, 2, 0, 2, 0, 1)$, $\quad J = \{2, 4, 6\}$

1. compute z:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_2 \\ z_4 \\ z_6 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 1, 0, -1, 0, 1, 0)$$

   not optimal; remove $k = 4$ from active set

2. compute Δx

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (-1, 0, 0)$$

3. minimum ratio test: $A\Delta x = (1, 0, 0, -1, 0, 0, -1)$

$$\hat{\alpha} = \mathrm{argmin}\{2/1\} = 2, \qquad j = 1$$

iteration 5: $x = (0, 0, 2)$, $b - Ax = (0, 0, 2, 2, 2, 0, 3)$, $J = \{1, 2, 6\}$

1. compute z:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_6 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (1, 1, 0, 0, 0, 1, 0)$$

   optimal

Degeneracy

- if x is degenerate, A_J has rank n but is not square
- if next point is degenerate, we have a tie in the minimization in step 3

solution

- define J to be a subset of n linearly independent active constraints
- A_J is square; steps 1 and 2 work as in the nondegenerate case
- in step 3, break ties arbitrarily

does it work?

- in step 3 we can have $\hat{\alpha} = 0$ (*i.e.*, x does not change)
- maybe this is okay, as long as J keeps changing

Example

$$
\begin{array}{ll}
\text{minimize} & -3x_1 + 5x_2 - x_3 + 2x_4 \\
\text{subject to} & \begin{bmatrix} 1 & -2 & -2 & 3 \\ 2 & -3 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}
$$

- $x = (0, 0, 0, 0)$ is a degenerate extreme point with

$$b - Ax = (0, 0, 1, 0, 0, 0, 0)$$

- start simplex with $J = \{4, 5, 6, 7\}$

iteration 1: $J = \{4, 5, 6, 7\}$

1. $z = (0, 0, 0, -3, 5, -1, 2)$: remove 4 from active set
2. $\Delta x = (1, 0, 0, 0)$
3. $A\Delta x = (1, 2, 0, -1, 0, 0, 0)$: $\hat{\alpha} = 0$, add 1 to active set

iteration 2: $J = \{1, 5, 6, 7\}$

1. $z = (3, 0, 0, 0, -1, -7, 11)$: remove 5 from active set
2. $\Delta x = (2, 1, 0, 0)$
3. $A\Delta x = (0, 1, 0, -2, -1, 0, 0)$: $\hat{\alpha} = 0$, add 2 to active set

iteration 3: $J = \{1, 2, 6, 7\}$

1. $z = (1, 1, 0, 0, 0, -4, 6)$: remove 6 from active set
2. $\Delta x = (-4, -3, 1, 0)$
3. $A\Delta x = (0, 0, 1, 4, 3, -1, 0)$: $\hat{\alpha} = 0$, add 4 to active set

iteration 4: $J = \{1, 2, 4, 7\}$

1. $z = (-2, 3, 0, 1, 0, 0, -1)$: remove 7 from active set
2. $\Delta x = (0, -1/4, 7/4, 1)$
3. $A\Delta x = (0, 0, 7/4, 0, 1/4, -7/4, -1)$: $\hat{\alpha} = 0$, add 5 to active set

iteration 5: $J = \{1, 2, 4, 5\}$

1. $z = (-1, 1, 0, -2, 4, 0, 0)$: remove 1 from active set
2. $\Delta x = (0, 0, -1, -1)$
3. $A\Delta x = (-1, 0, -1, 0, 0, 1, 1)$: $\hat{\alpha} = 0$, add 6 to active set

iteration 6: $J = \{2, 4, 5, 6\}$

1. $z = (0, -2, 0, -7, 11, 1, 0)$: remove 2 from active set
2. $\Delta x = (0, 0, 0, -1)$
3. $A\Delta x = (-3, -1, 0, 0, 0, 0, 1)$: $\hat{\alpha} = 0$, add 7 to active set

iteration 7: $J = \{4, 5, 6, 7\}$, the initial active set

Bland's ('least-index') pivoting rule

no cycling occurs if we follow the following rule

- in step 2, choose the smallest k for which $z_k < 0$
- if there is a tie in step 3, choose the smallest j

proof (by contradiction) suppose there is a cycle, *i.e.*, for some $q > p$

$$x^{(p)} = x^{(p+1)} = \cdots = x^{(q)}, \qquad J^{(p)} \neq J^{(p+1)} \neq \cdots \neq J^{(q)} = J^{(p)}$$

where $x^{(s)}$, $J^{(s)}$, $z^{(s)}$, $\Delta x^{(s)}$ are the values of x, J, z, Δx at iteration s

we also define

- k_s: index removed from $J^{(s)}$ in iteration s; j_s: index added in iteration s
- $\bar{k} = \max_{p \le s \le q-1} k_s$
- r: the iteration ($p \le r \le q-1$) in which $\bar{k}$ is removed ($\bar{k} = k_r$)
- t: the iteration ($r < t \le q$) in which $\bar{k}$ is added back again ($\bar{k} = j_t$)

at iteration r we remove index $\bar{k}$ from $J^{(r)}$; therefore

- $z_{\bar{k}}^{(r)} < 0$
- $z_i^{(r)} \geq 0$ for $i \in J^{(r)}$, $i < \bar{k}$ (otherwise we should have removed i)
- $z_i^{(r)} = 0$ for $i \notin J^{(r)}$ (by definition of $z^{(r)}$)

at iteration t we add index $\bar{k}$ to $J^{(t)}$; therefore

- $a_{\bar{k}}^T \Delta x^{(t)} > 0$
- $a_i^T \Delta x^{(t)} \leq 0$ for $i \in J^{(r)}$, $i < \bar{k}$

  (otherwise we should have added i, since $b_i - a_i^T x = 0$ for all $i \in J^{(r)}$)
- $a_i^T \Delta x^{(t)} = 0$ for $i \in J^{(r)}$, $i > \bar{k}$

  (if $i > \bar{k}$ and $i \in J^{(r)}$ then it is never removed, so $i \in J^{(t)} \setminus \{k_t\}$)

conclusion: ${z^{(r)}}^T A \Delta x^{(t)} < 0$

a contradiction, because $-{z^{(r)}}^T A \Delta x^{(t)} = c^T \Delta x^{(t)} \leq 0$

Example

example of page 12–21, same starting point but applying Bland's rule

iteration 1: $J = \{4, 5, 6, 7\}$

1. $z = (0, 0, 0, -3, 5, -1, 2)$: remove 4 from active set
2. $\Delta x = (1, 0, 0, 0)$
3. $A\Delta x = (1, 2, 0, -1, 0, 0, 0)$: $\hat{\alpha} = 0$, add 1 to active set

iteration 2: $J = \{1, 5, 6, 7\}$

1. $z = (3, 0, 0, 0, -1, -7, 11)$: remove 5 from active set
2. $\Delta x = (2, 1, 0, 0)$
3. $A\Delta x = (0, 1, 0, -2, -1, 0, 0)$: $\hat{\alpha} = 0$, add 2 to active set

iteration 3: $J = \{1, 2, 6, 7\}$

1. $z = (1, 1, 0, 0, 0, -4, 6)$: remove 6 from active set
2. $\Delta x = (-4, -3, 1, 0)$
3. $A\Delta x = (0, 0, 1, 4, 3, -1, 0)$: $\hat{\alpha} = 0$, add 4 to active set

iteration 4: $J = \{1, 2, 4, 7\}$

1. $z = (-2, 3, 0, 1, 0, 0, -1)$: remove 1 from active set
2. $\Delta x = (0, -1/4, 3/4, 1)$
3. $A\Delta x = (-1, 0, 3/4, 0, 1/4, -3/4, 0)$: $\hat{\alpha} = 0$, add 5 to active set

iteration 5: $J = \{2, 4, 5, 7\}$

1. $z = (0, -1, 0, -5, 8, 0, 1)$: remove 2 from active set
2. $\Delta x = (0, 0, 1, 0)$
3. $A\Delta x = (-2, -1, 1, 0, 0, -1, 0)$: $\hat{\alpha} = 1$, add 3 to active set

new $x = (0, 0, 1, 0)$, $b - Ax = (2, 1, 0, 0, 0, 1, 0)$

iteration 6: $J = \{3, 4, 5, 7\}$

1. $z = (0, 0, 1, -3, 5, 0, 2)$: remove 4 from active set
2. $\Delta x = (1, 0, 0, 0)$
3. $A\Delta x = (1, 2, 0, -1, 0, 0, 0)$: $\hat{\alpha} = 1/2$, add 2 to active set

new $x = (1/2, 0, 1, 0)$, $b - Ax = (3/2, 0, 0, 1/2, 0, 1, 0)$

iteration 7: $J = \{1, 3, 5, 7\}$

1. $z = (3, 0, 7, 0, -1, 0, 11)$: remove 5 from active set
2. $\Delta x = (2, 1, 0, 0)$
3. $A\Delta x = (0, 1, 0, -2, -1, 0, 0)$: $\hat{\alpha} = 0$, add 2 to active set

iteration 8: $J = \{1, 2, 3, 7\}$

1. $z = (1, 1, 4, 0, 0, 0, 6)$: optimal

Initialization

linear program with variable bounds

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b, \quad x \geq 0 \end{array}$$

(general: free x_k can be split as $x_k = x_k^+ - x_k^-$ with $x_k^+ \geq 0$, $x_k^- \geq 0$)

phase I problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax \leq (1-t)b, \quad x \geq 0, \quad 0 \leq t \leq 1 \end{array}$$

- $x = 0$, $t = 1$ is an extreme point for phase I problem
- can compute an optimal extreme point $x^\star$, $t^\star$ via simplex method
- if $t^\star > 0$, original problem is infeasible
- if $t^\star = 0$, then $x^\star$ is an extreme point of original problem

Implementation

- most expensive step: solution of two sets of linear equations

$$A_J^T z_J = -c, \qquad A_J \Delta x = -(e_k)_J$$

  where e_k is kth unit vector
- one row of A_J changes at each iteration

efficient implementation: propagate LU factorization of A_J

- given the factorization, the equations can be solved in $O(n^2)$ operations
- updating LU factorization after changing a row costs $O(n^2)$ operations

total cost is $O(n^2)$ per iteration (and much less than $O(n^2)$ if A is sparse)

Complexity

worst-case

- for most pivoting rules, there exist examples where the number of iterations grows exponentially with n and m
- it is an open question whether there exists a pivoting rule for which the number of iterations is bounded by a polynomial of n and m

in practice: very efficient (#iterations typically grows linearly with m, n)