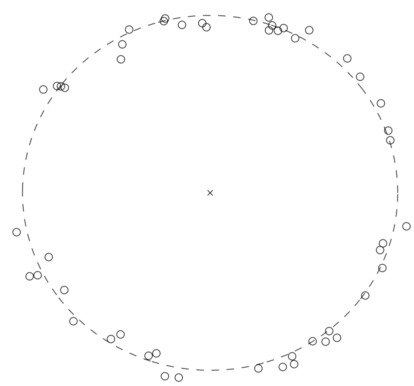
Homework 1

1. *Least squares fit of a circle to points.* In this problem we use least squares to fit a circle to given points (u_i, v_i) in a plane, as shown in the figure.



The variables (u_c, v_c) denote the center of the circle and R its radius. A point (u, v) is on the circle if $(u - u_c)^2 + (v - v_c)^2 = R^2$. We formulate the fitting problem as an optimization problem

$$\text{minimize} \quad \sum_{i=1}^{m} \left((u_i - u_c)^2 + (v_i - v_c)^2 - R^2\right)^2$$

with three variables u_c, v_c, R.

(a) Show that the problem can be written as a linear least squares problem

$$\text{minimize } \|Ax - b\|_2^2 \tag{1}$$

if we make a change of variables and use as variables

$$x_1 = u_c, \qquad x_2 = v_c, \qquad x_3 = u_c^2 + v_c^2 - R^2.$$

(b) Use the normal equations (optimality conditions) $A^TAx = A^Tb$ of the least squares problem to show that the optimal solution $\hat{x}$ of the least squares problem satisfies

$$\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3 \geq 0.$$

This is necessary to compute $R = \sqrt{\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3}$ from $\hat{x}$.

(c) Test your formulation on the problem data in the file `circlefit.m` on the course website. The commands

```
circlefit;
plot(u, v, 'o');
axis equal
```

will create a plot of the $m = 50$ points (u_i, v_i) in the figure. Use the MATLAB command `x = A \ b` to solve the least squares problem (1).

2. Let X be a symmetric matrix partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}. \tag{2}$$

If A is nonsingular, the matrix $S = C - B^T A^{-1} B$ is called the *Schur complement* of A in X. It can be shown that if A is positive definite, then $X \succeq 0$ (X is positive semidefinite) if and only if $S \succeq 0$ (see page 650 of the textbook). In this exercise we prove the extension of this result to singular A mentioned on page 651 of the textbook.

(a) Suppose $A = 0$ in (2). Show that $X \succeq 0$ if and only if $B = 0$ and $C \succeq 0$.

(b) Let A be a symmetric $n \times n$ matrix with eigenvalue decomposition

$$A = Q\Lambda Q^T,$$

where Q is orthogonal ($Q^T Q = QQ^T = I$) and $\Lambda = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Assume the first r eigenvalues λ_i are nonzero and $\lambda_{r+1} = \cdots = \lambda_n = 0$. Partition Q and Λ as

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with Q_1 of size $n \times r$, Q_2 of size $n \times (n - r)$, and $\Lambda_1 = \mathbf{diag}(\lambda_1, \ldots, \lambda_r)$. The matrix

$$A^\dagger = Q_1 \Lambda_1^{-1} Q_1^T$$

is called the *pseudo-inverse* of A.

Verify that

$$AA^\dagger = A^\dagger A = Q_1 Q_1^T, \qquad I - AA^\dagger = I - A^\dagger A = Q_2 Q_2^T.$$

The matrix-vector product $AA^\dagger x = Q_1 Q_1^T x$ is the orthogonal projection of the vector x on the range of A. The matrix-vector product $(I - AA^\dagger)x = Q_2 Q_2^T x$ is the projection on the nullspace.

(c) Show that the block matrix X in (2) is positive semidefinite if and only if

$$A \succeq 0, \qquad (I - AA^\dagger)B = 0, \qquad C - B^T A^\dagger B \succeq 0.$$

(The second condition means that the columns of B are in the range of A.)

Hint. Let $A = Q\Lambda Q^T$ be the eigenvalue decomposition of A. The matrix X in (2) is positive semidefinite if and only if the matrix

$$\begin{bmatrix} Q^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Lambda & Q^T B \\ B^T Q & C \end{bmatrix}$$

is positive semidefinite. Use the observation in part (a) and the Schur complement characterization for positive definite 2×2 block matrices to show the result.

3. This problem is an introduction to the MATLAB software package CVX that will be used in the course. CVX can be downloaded from `www.cvxr.com`.

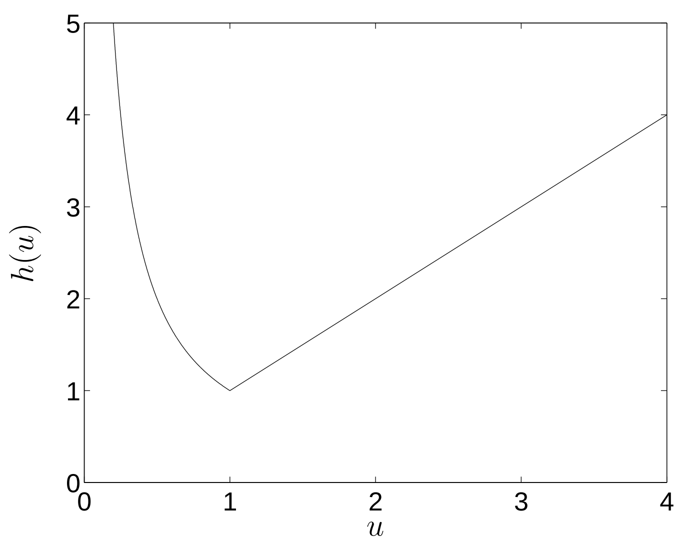
   We consider the illumination problem of lecture 1. We take $I_{\mathrm{des}} = 1$ and $p_{\max} = 1$, so the problem is

$$\begin{array}{ll} \text{minimize} & f_0(p) = \max\limits_{k=1,\ldots,n} |\log(a_k^T p)| \\ \text{subject to} & 0 \leq p_j \leq 1, \quad j = 1, \ldots, m, \end{array} \tag{3}$$

   with variable $p \in \mathbf{R}^m$. As mentioned in the lecture, the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \max\limits_{k=1,\ldots,n} h(a_k^T p) \\ \text{subject to} & 0 \leq p_j \leq 1, \quad j = 1, \ldots, m, \end{array} \tag{4}$$

   where $h(u) = \max\{u, 1/u\}$ for $u > 0$. The function h, shown in the figure below, is nonlinear, nondifferentiable, and convex.



   To see the equivalence between (3) and (4), we note that

$$f_0(p) = \max_{k=1,\ldots,n} |\log(a_k^T p)|$$

$$
\begin{aligned}
&= \max_{k=1,\ldots,n} \max\left\{\log(a_k^T p), \log(1/a_k^T p)\right\} \\
&= \log \max_{k=1,\ldots,n} \max\left\{a_k^T p, 1/a_k^T p\right\} \\
&= \log \max_{k=1,\ldots,n} h(a_k^T p),
\end{aligned}
$$

and since the logarithm is a monotonically increasing function, minimizing f_0 is equivalent to minimizing $\max_{k=1,\ldots,n} h(a_k^T p)$.

The problem data are given in the file `illum_data.m` posted on the course website. Executing this file in MATLAB creates the $n \times m$-matrix A (which has rows a_k^T). There are 10 lamps ($m = 10$) and 20 patches ($n = 20$).

Use the following methods to compute four approximate solutions and the exact solution, and compare the answers (the vectors p and the corresponding values of $f_0(p)$).

(a) *Equal lamp powers.* Take $p_j = \gamma$ for $j = 1, \ldots, m$. Plot $f_0(p)$ versus γ over the interval $[0, 1]$. Graphically determine the optimal value of γ, and the associated objective value. The objective function $f_0(p)$ can be evaluated in MATLAB as `max(abs(log(A*p)))`.

(b) *Least-squares with saturation.* Solve the least squares problem

$$
\text{minimize} \quad \sum_{k=1}^{n} (a_k^T p - 1)^2 = \|Ap - \mathbf{1}\|_2^2.
$$

If the solution has negative coefficients, set them to zero; if some coefficients are greater than 1, set them to 1. Use the MATLAB command `x = A \ b` to solve a least squares problem (minimize $\|Ax - b\|_2^2$).

(c) *Regularized least squares.* Solve the regularized least squares problem

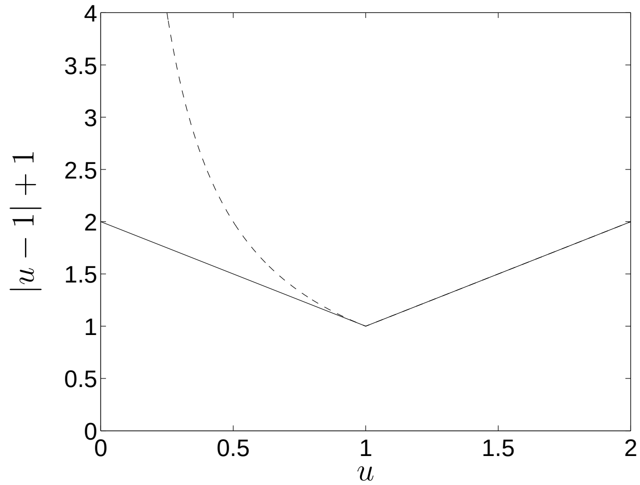
$$
\text{minimize} \quad \sum_{k=1}^{n} (a_k^T p - 1)^2 + \rho \sum_{j=1}^{m} (p_j - 0.5)^2 = \|Ap - \mathbf{1}\|_2^2 + \rho \left\|p - (1/2)\mathbf{1}\right\|_2^2,
$$

where $\rho > 0$ is a parameter. Increase ρ until all coefficients of p are in the interval $[0, 1]$.

(d) *Chebyshev approximation.* Solve the problem

$$
\begin{array}{ll}
\text{minimize} & \max_{k=1,\ldots,n} |a_k^T p - 1| = \|Ap - \mathbf{1}\|_\infty \\
\text{subject to} & 0 \le p_j \le 1, \quad j = 1, \ldots, m.
\end{array}
$$

We can think of this problem as obtained by approximating the nonlinear function $h(u)$ by a piecewise-linear function $|u - 1| + 1$. As shown in the figure below, this is a good approximation around $u = 1$. This problem can be converted to a linear program and solved using the MATLAB function `linprog`. It can also be solved directly in CVX, using the expression `norm(A*p - 1, inf)` to specify the cost function.



(e) *Exact solution.* Finally, use CVX to solve

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,n} \max(a_k^T p, 1/a_k^T p) \\ \text{subject to} & 0 \leq p_j \leq 1, \quad j = 1, \ldots, m. \end{array}$$

Use the CVX function `inv_pos()` to expresss the function $f(x) = 1/x$ with domain $\mathbf{R}_{++}$.