Homework 2

1. *Helly's theorem.*

   (a) (Radon's theorem) Let $X = \{x_1, \ldots, x_m\}$ be a set of m points in $\mathbf{R}^n$, where $m \geq n+2$. Show that X can be partitioned in two sets S and $T = X \setminus S$ such that

   $$\mathbf{conv}\, S \cap \mathbf{conv}\, T \neq \emptyset.$$

   Here $\mathbf{conv}\, S$ and $\mathbf{conv}\, T$ denote the convex hulls of S and T.

   Hint. Since $m \geq n+2$, the vectors $(x_i, 1)$, $i = 1, \ldots, m$, are linearly dependent. Therefore there exists a nonzero y such that

   $$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = 0.$$

   Use y to define S and T, and to construct a point $x \in \mathbf{conv}\, S \cap \mathbf{conv}\, T$.

   (b) Use the result in (a) to prove the following. Let $S_1, \ldots, S_m$ be a collection of convex sets in $\mathbf{R}^n$, where $m \geq n+2$. Suppose the intersection of every $m-1$ sets from the collection is nonempty, *i.e.*, the set

   $$\bigcap_{i \in \{1,\ldots,m\} \setminus \{k\}} S_i = S_1 \cap \cdots \cap S_{k-1} \cap S_{k+1} \cap \cdots \cap S_m$$

   is nonempty for each $k = 1, \ldots, m$. Then the intersection of all sets $S_1, \ldots, S_m$ is nonempty:

   $$\bigcap_{i=1,\ldots,m} S_i = S_1 \cap \cdots \cap S_m \neq \emptyset.$$

   Hint. Apply the result in part (a) to m points $x_1, \ldots, x_m$ chosen to satisfy

   $$x_k \in \bigcap_{i \in \{1,\ldots,m\} \setminus \{k\}} S_i.$$

The result in (b) is easily rephrased in a more general form, known as Helly's theorem. Let $S_1, \ldots, S_m$ be a collection of m convex sets in $\mathbf{R}^n$. Suppose the intersection of every $k \leq n+1$ sets from the collection is nonempty. Then the intersection of all sets $S_1, \ldots, S_m$ is nonempty.

2. Exercise T3.2.
3. Exercise T3.18 (a).
4. Exercise A2.10.
5. Exercise A5.8.