Homework 2

Submit answers for problems 1–4 only.

1. Exercise T2.12 (d,g).

2. Polar of a set. The polar of a set $C \subseteq \mathbb{R}^n$ is defined as
   \[ C^o = \{ y \in \mathbb{R}^n \mid y^T x \leq 1 \text{ for all } x \in C \}. \]
   (a) Show that $C^o$ is convex (even if $C$ is not).
   (b) What is the polar of a cone?
   (c) What are the polars of the following three sets?
      \[ C_1 = \{ x \mid \| x \|_2 \leq 1 \}, \quad C_2 = \{ x \mid \| x \|_1 \leq 1 \}, \quad C_3 = \{ x \mid 1^T x = 1, \ x \succeq 0 \}. \]
      (Here $1$ denotes the vector of ones.)

3. Exercise A2.10.

4. Exercise A5.8.

Problems 5–8 are additional practice problems and will not be graded.

5. Exercise T2.37.

6. Polar of a convex set. Suppose $C$ is closed, convex, and $0 \in C$. Show that $(C^o)^o = C$.
   (a) By definition of $C^o$, we have
   \[ y^T x \leq 1 \text{ for all } x \in C, \ y \in C^o. \]
   Show that this implies that $C \subseteq (C^o)^o$ (without assumptions on $C$).
   (b) Now suppose $C$ is closed and convex, with $0 \in C$. Show that $(C^o)^o \subseteq C$. (Combined with the result of part (a), this proves that $(C^o)^o = C$.)
   \text{Hint.} Show that if $x \not\in C$, then $x \not\in (C^o)^o$. To do this you can apply the strict separating hyperplane theorem of page 49 of the textbook: If $C$ is a closed convex set and $x \not\in C$, then there exists a vector $a \neq 0$ and a scalar $b$ such that
   \[ a^T x > b, \quad a^T z \leq b \text{ for all } z \in C. \]

7. Exercise T3.1.

8. Exercise T3.18 (a).