

### Additional problems for homework #5

1. *Generalized posynomials.* In lecture 4 we defined a *posynomial* as a function of the form

$$f(x_1, \dots, x_n) = \sum_{i=1}^m \beta_i \prod_{j=1, \dots, n} x_j^{a_{ij}}, \quad \text{dom } f = \mathbf{R}_{++}^n,$$

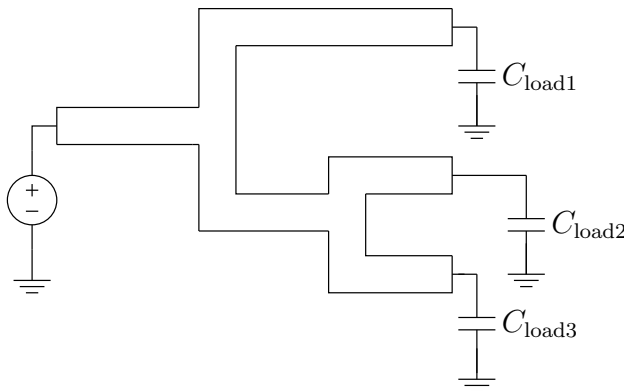
with  $\beta_i > 0$ . The exponents  $a_{ij}$  can take any real value. Let us call the posynomial *monotone* if the exponents  $a_{ij}$  are all nonnegative.

A *generalized posynomial* is a function that can be constructed by repeated application of the following rules.

- (a) A posynomial is a generalized posynomial.
- (b) The maximum  $f(x) = \max \{g_1(x), \dots, g_m(x)\}$  of  $m$  generalized posynomials  $g_k$  is a generalized posynomial.
- (c) The composition  $f(x) = h(g_1(x), \dots, g_m(x))$  of a *monotone* posynomial  $h$  of  $m$  variables with generalized posynomials  $g_1, \dots, g_m$  is a generalized posynomial.

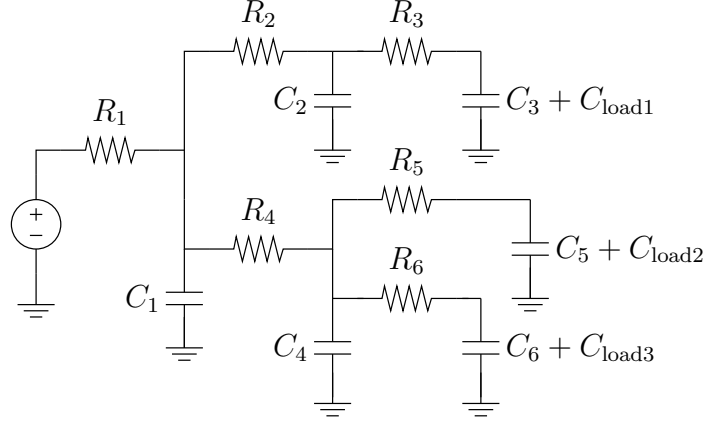
Show that if  $f(x_1, \dots, x_n)$  is a generalized posynomial, then the function  $\tilde{f}(y_1, \dots, y_n) = \log f(e^{y_1}, \dots, e^{y_n})$  is convex. In other words, show that rules 2 and 3 listed above preserve this property: if the functions  $\log g_k(e^{y_1}, \dots, e^{y_n})$  are convex, then a function  $f$  constructed using rules 2 or 3 has the property that  $\log f(e^{y_1}, \dots, e^{y_n})$  is convex.

2. *Interconnect sizing.* We consider the problem of sizing the interconnecting wires of the simple circuit shown below, in which one voltage source drives three different capacitive loads  $C_{\text{load1}}$ ,  $C_{\text{load2}}$ , and  $C_{\text{load3}}$ .



We divide the wires into 6 segments of fixed length  $l_i$ ; the optimization variables will be the widths  $w_i$  of the segments. (The height of the wires is related to the particular integrated circuit technology process, and is fixed.) We take the lengths  $l_i$  to be one, for simplicity.

In the next figure each of the wire segments is modeled by a simple RC circuit, with the resistance inversely proportional to the width of the segment and the capacitance proportional to the width.



The capacitance and resistance of the  $i$ th segment is thus

$$C_i = k_0 w_i, \quad R_i = \rho / w_i, \quad i = 1, \dots, 6,$$

where  $k_0$  and  $\rho$  are positive constants, which we take to be one for simplicity. We also take  $C_{\text{load1}} = 1.5$ ,  $C_{\text{load2}} = 1$ , and  $C_{\text{load3}} = 5$ .

We are interested in the trade-off between area and delay. The total area used by the wires is, of course,

$$A = \sum_{i=1}^6 w_i l_i = \sum_{i=1}^6 w_i.$$

We will use the Elmore delay to model the delay from the source to each of the loads. The Elmore delays to loads 1, 2, and 3 are defined as

$$\begin{aligned} T_1 &= (C_3 + C_{\text{load1}})(R_1 + R_2 + R_3) + C_2(R_1 + R_2) \\ &\quad + (C_1 + C_4 + C_5 + C_6 + C_{\text{load2}} + C_{\text{load3}})R_1 \\ T_2 &= (C_5 + C_{\text{load2}})(R_1 + R_4 + R_5) + C_4(R_1 + R_4) \\ &\quad + (C_6 + C_{\text{load3}})(R_1 + R_4) + (C_1 + C_2 + C_3 + C_{\text{load1}})R_1 \\ T_3 &= (C_6 + C_{\text{load3}})(R_1 + R_4 + R_6) + C_4(R_1 + R_4) \\ &\quad + (C_1 + C_2 + C_3 + C_{\text{load1}})R_1 + (C_5 + C_{\text{load2}})(R_1 + R_4). \end{aligned}$$

Our main interest is in the maximum of these delays,

$$T = \max \{T_1, T_2, T_3\}.$$

We also impose minimum and maximum allowable values for the wire widths:

$$W_{\min} \leq w_i \leq W_{\max}, \quad i = 1, \dots, 6.$$

For our specific problem, we take  $W_{\min} = 0.1$  and  $W_{\max} = 10$ .

We compare two choices of wire widths.

- (a) *Equal wire widths.* Plot the values of area  $A$  versus delay  $T$ , obtained if you take equal wire widths  $w_i$  (varying between  $W_{\min}$  and  $W_{\max}$ ).
- (b) *Optimal wire widths.* The optimal area-delay trade-off curve can be computed by scalarization, *i.e.*, by minimizing  $A + \mu T$ , subject to the constraints on  $w$ , for a large number of different positive values of  $\mu$ . Verify that the scalarized problem is a geometric program (GP). For the specific problem parameters given, compute the area-delay trade-off curve using CVX. You can choose the values of  $\mu$  logarithmically spaced between  $10^{-3}$  and  $10^3$ . Consult chapter 7 of the CVX user guide for details on how to solve GPs. (For reasons explained in the user guide, CVX is not very fast when solving GPs. If needed, you can limit the number of weights  $\mu$ , for example, to 10 or 20.)

Compare the optimal trade-off curve with the one obtained in part (a).

3. Let  $X$  be a symmetric matrix partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}. \quad (1)$$

If  $A$  is nonsingular, the matrix  $S = C - B^T A^{-1} B$  is called the *Schur complement* of  $A$  in  $X$ . It can be shown that if  $A$  is positive definite, then  $X \succeq 0$  ( $X$  is positive semidefinite) if and only if  $S \succeq 0$  (see page 650 of the textbook). In this exercise we prove the extension of this result to singular  $A$  mentioned on page 651 of the textbook.

- (a) Suppose  $A = 0$  in (1). Show that  $X \succeq 0$  if and only if  $B = 0$  and  $C \succeq 0$ .
- (b) Let  $A$  be a symmetric  $n \times n$  matrix with eigenvalue decomposition

$$A = Q\Lambda Q^T,$$

where  $Q$  is orthogonal ( $Q^T Q = Q Q^T = I$ ) and  $\Lambda = \mathbf{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Assume the first  $r$  eigenvalues  $\lambda_i$  are nonzero and  $\lambda_{r+1} = \dots = \lambda_n = 0$ . Partition  $Q$  and  $\Lambda$  as

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with  $Q_1$  of size  $n \times r$ ,  $Q_2$  of size  $n \times (n - r)$ , and  $\Lambda_1 = \mathbf{diag}(\lambda_1, \dots, \lambda_r)$ . The matrix

$$A^\dagger = Q_1 \Lambda_1^{-1} Q_1^T$$

is called the *pseudo-inverse* of  $A$ . Verify that

$$AA^\dagger = A^\dagger A = Q_1 Q_1^T, \quad I - AA^\dagger = I - A^\dagger A = Q_2 Q_2^T.$$

The matrix-vector product  $AA^\dagger x = Q_1 Q_1^T x$  is the orthogonal projection of the vector  $x$  on the range of  $A$ . The matrix-vector product  $(I - AA^\dagger)x = Q_2 Q_2^T x$  is the projection on the nullspace.

(c) Show that the block matrix  $X$  in (1) is positive semidefinite if and only if

$$A \succeq 0, \quad (I - AA^\dagger)B = 0, \quad C - B^T A^\dagger B \succeq 0.$$

(The second condition means that the columns of  $B$  are in the range of  $A$ .)

*Hint.* Let  $A = Q\Lambda Q^T$  be the eigenvalue decomposition of  $A$ . The matrix  $X$  in (1) is positive semidefinite if and only if the matrix

$$\begin{bmatrix} Q^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Lambda & Q^T B \\ B^T Q & C \end{bmatrix}$$

is positive semidefinite. Use the observation in part (a) and the Schur complement characterization for positive definite  $2 \times 2$  block matrices to show the result.