Additional problem for assignment #6

Consider the optimization problem

\[
\text{minimize } \|Ax - b\|_2 + \gamma \|x\|_1
\]

with \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \) and \( \gamma > 0. \) The variable is an \( n \)-vector \( x. \)

1. Derive the Lagrange dual of the equivalent problem

\[
\text{minimize } \|y\|_2 + \gamma \|x\|_1 \\
\text{subject to } Ax - b = y
\]

with variables \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m. \)

2. Suppose \( Ax^* - b \neq 0 \) where \( x^* \) is an optimal point. Define \( r = (Ax^* - b)/\|Ax^* - b\|_2. \)

Show that

\[
\|A^T r\|_\infty \leq \gamma, \quad r^T Ax^* + \gamma \|x^*\|_1 = 0.
\]

3. Show that if the Euclidean norm of the \( i \)-th column of \( A \) is less than \( \gamma \), then \( x^*_i = 0. \)