11. Interior-point methods

- inequality constrained minimization
- logarithmic barrier function and central path
- barrier method
- feasibility and phase I methods
- complexity analysis via self-concordance
- second-order cone and semidefinite programming

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{1}$$

- f_i convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained
- we assume the problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \qquad A\tilde{x} = b$$

hence, strong duality holds and dual optimum is attained

Unconstrained (or equality-constrained) approximation

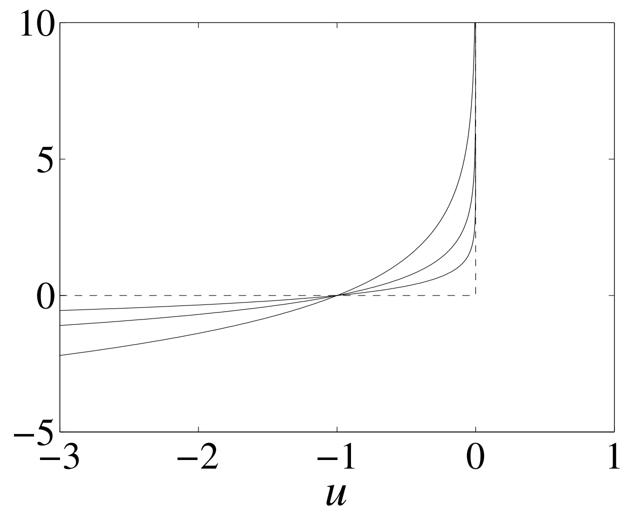
- write (1) as problem without inequality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m h(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

  where h is indicator function of $\mathbf{R}_-$: $h(u) = 0$ if $u \leq 0$ and $h(u) = \infty$ otherwise

- approximate indicator function by logarithmic barrier:

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$



- an equality constrained problem
- $t > 0$, approximation improves as $t \to \infty$

Logarithmic barrier function

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \qquad \mathbf{dom}\,\phi = \{x \mid f_1(x) < 0, \ldots, f_m(x) < 0\}$$

- a convex function (follows from composition rules)
- twice continuously differentiable, with derivatives

$$\begin{aligned} \nabla\phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\ \nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x) \end{aligned}$$

Central path

- for $t > 0$, define $x^\star(t)$ as the solution of the *centering problem*

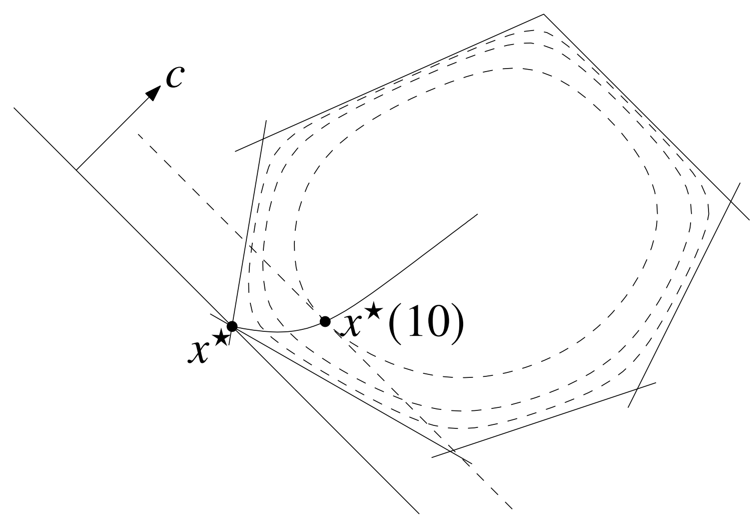
$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

  (for now, assume $x^\star(t)$ exists and is unique for each $t > 0$)

- the set $\{x^\star(t) \mid t > 0\}$ is called the *central path*

Example: central path for an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, 6 \end{array}$$



hyperplane $c^T x = c^T x^\star(t)$ is tangent to level curve of ϕ through $x^\star(t)$

Dual points on central path

- optimality condition for centering problem: $Ax = b$ and there exists a w such that

$$\begin{aligned} 0 &= t\nabla f_0(x) + \nabla\phi(x) + A^T w \\ &= t\nabla f_0(x) + \sum_{i=1}^m \frac{1}{-f_i(x)}\nabla f_i(x) + A^T w \end{aligned}$$

- point on central path $x^\star(t)$ minimizes the Lagrangian of the original problem

$$L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \nu^T(Ax - b)$$

for λ, ν given by

$$\lambda_i^\star(t) = \frac{1}{-t f_i(x^\star(t))}, \quad i = 1,\ldots,m, \qquad \nu^\star(t) = w/t$$

centering gives a strictly primal feasible $x^\star(t)$ *and* a dual feasible $\lambda^\star(t), \nu^\star(t)$

Duality gap on central path

- value of dual objective function at $\lambda^\star(t)$, $\nu^\star(t)$ is

$$
\begin{aligned}
g(\lambda^\star(t), \nu^\star(t)) &= \inf_x L(x, \lambda^\star(t), \nu^\star(t)) \\
&= L(x^\star(t), \lambda^\star(t), \nu^\star(t)) \\
&= f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star(t) f_i(x^\star(t)) + {\nu^\star}^T (Ax^\star - b) \\
&= f_0(x^\star(t)) - \frac{m}{t}
\end{aligned}
$$

- this confirms the intuitive idea that $f_0(x^\star(t)) \to p^\star$ if $t \to \infty$:

$$
f_0(x^\star(t)) - p^\star \le \frac{m}{t}
$$

Interpretation via KKT conditions

$x = x^\star(t)$, $\lambda = \lambda^\star(t)$, $\nu = \nu^\star(t)$ satisfy

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $Ax = b$
2. dual inequality: $\lambda \succeq 0$
3. approximate complementary slackness:

$$\lambda_i f_i(x) = -\frac{1}{t}, \quad i = 1, \ldots, m$$

4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0$$

difference with KKT conditions is that condition 3 replaces $\lambda_i f_i(x) = 0$

Force field interpretation

Centering problem (for problem with no equality constraints)

$$\text{minimize} \quad t f_0(x) - \sum_{i=1}^{m} \log(-f_i(x))$$

Force field interpretation

- $t f_0(x)$ is potential of force field

$$F_0(x) = -t \nabla f_0(x)$$

- $-\log(-f_i(x))$ is potential of force field

$$F_i(x) = (1/f_i(x)) \nabla f_i(x)$$

- the forces balance at $x^\star(t)$:

$$F_0(x^\star(t)) + \sum_{i=1}^{m} F_i(x^\star(t)) = 0$$

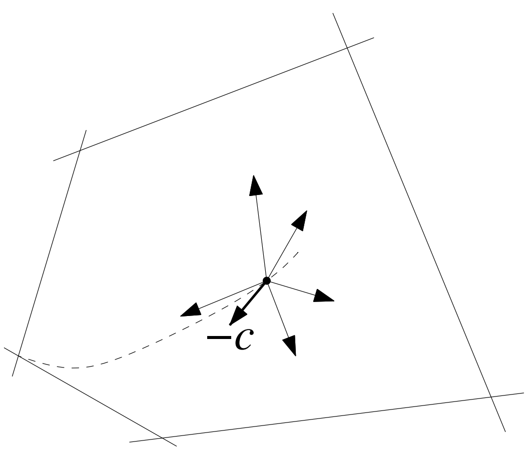
Example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m \end{array}$$

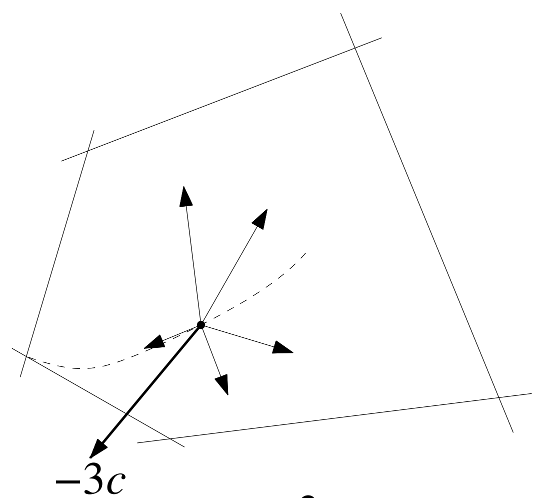
- objective force field is constant: $F_0(x) = -tc$
- constraint force field decays as inverse distance to constraint hyperplane:

$$F_i(x) = \frac{-a_i}{b_i - a_i^T x}, \qquad \|F_i(x)\|_2 = \frac{1}{d(x, \mathcal{H}_i)}$$

where $d(x, \mathcal{H}_i)$ is distance of x to hyperplane $\mathcal{H}_i = \{x \mid a_i^T x = b_i\}$



$t = 1$



$t = 3$

Barrier method

given: strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$

repeat

1. *centering step:* compute $x^\star(t)$ by minimizing $t f_0(x) + \phi(x)$ subject to $Ax = b$
2. *update:* $x := x^\star(t)$
3. *stopping criterion:* quit if $m/t < \epsilon$
4. *increase* t*:* $t := \mu t$

- terminates with strictly feasible point that satisfies $f_0(x) - p^\star \leq m/t < \epsilon$
- centering is usually done using Newton's method, starting at current x
- an outer iteration loop (steps 1–4) and an inner (Newton) iteration loop (step 1)
- choice of μ involves trade-off between number of outer and inner iterations
- typical values of μ are 10–20
- several heuristics exist for choosing $t^{(0)}$

Convergence analysis

Number of outer (centering) iterations: exactly

$$\left\lceil \frac{\log(m/(\epsilon t^{(0)}))}{\log \mu} \right\rceil$$

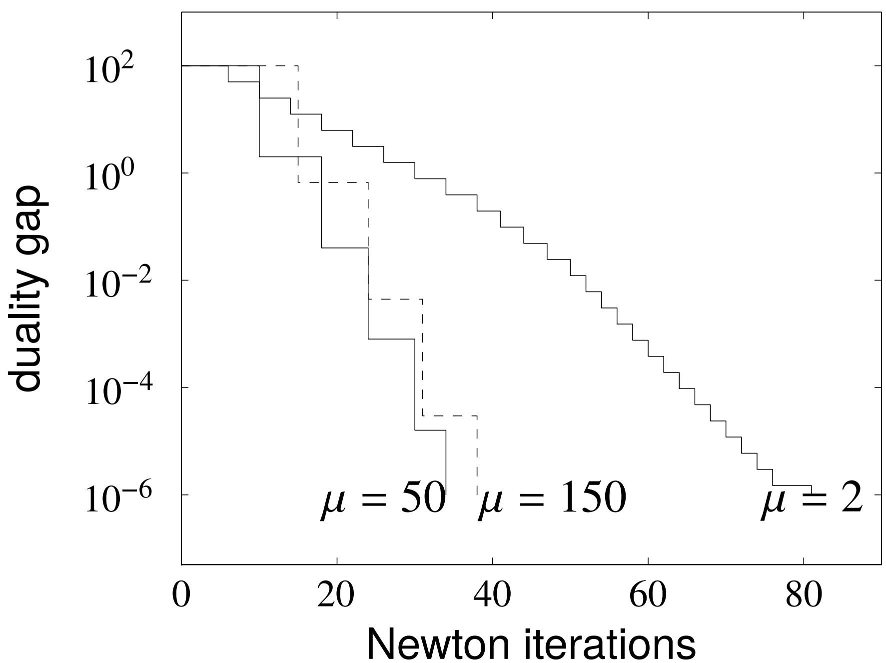
plus the initial centering step (to compute $x^\star(t^{(0)})$)

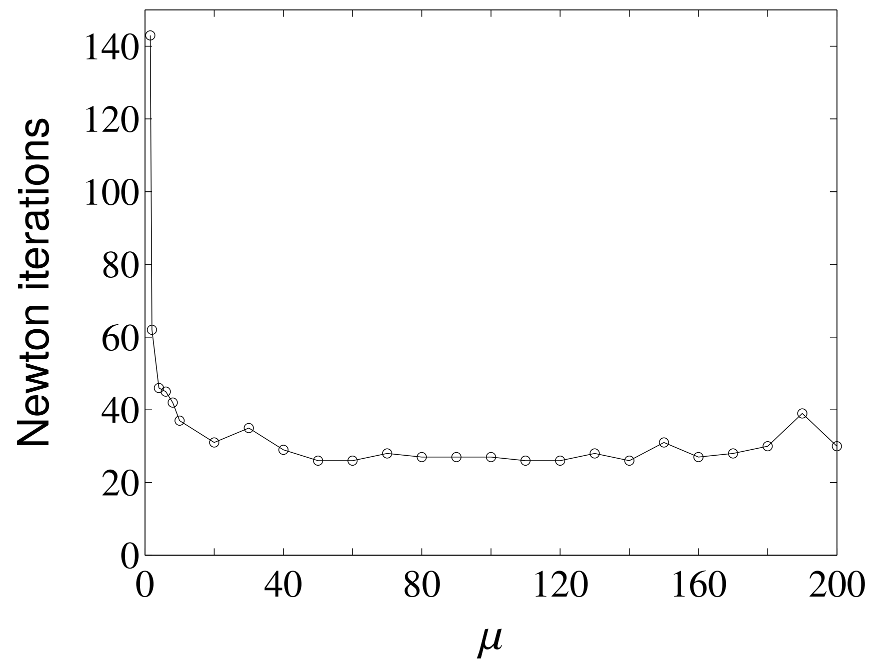
Centering problem: see convergence analysis of Newton's method

- $t f_0 + \phi$ must have closed sublevel sets for $t \geq t^{(0)}$
- classical analysis requires strong convexity, Lipschitz continuity of Hessian
- analysis via self-concordance requires self-concordance of $t f_0 + \phi$
- the additional assumptions also guarantee that solution exists and is unique

Example: inequality form LP

LP with $m = 100$ inequalities, $n = 50$ variables



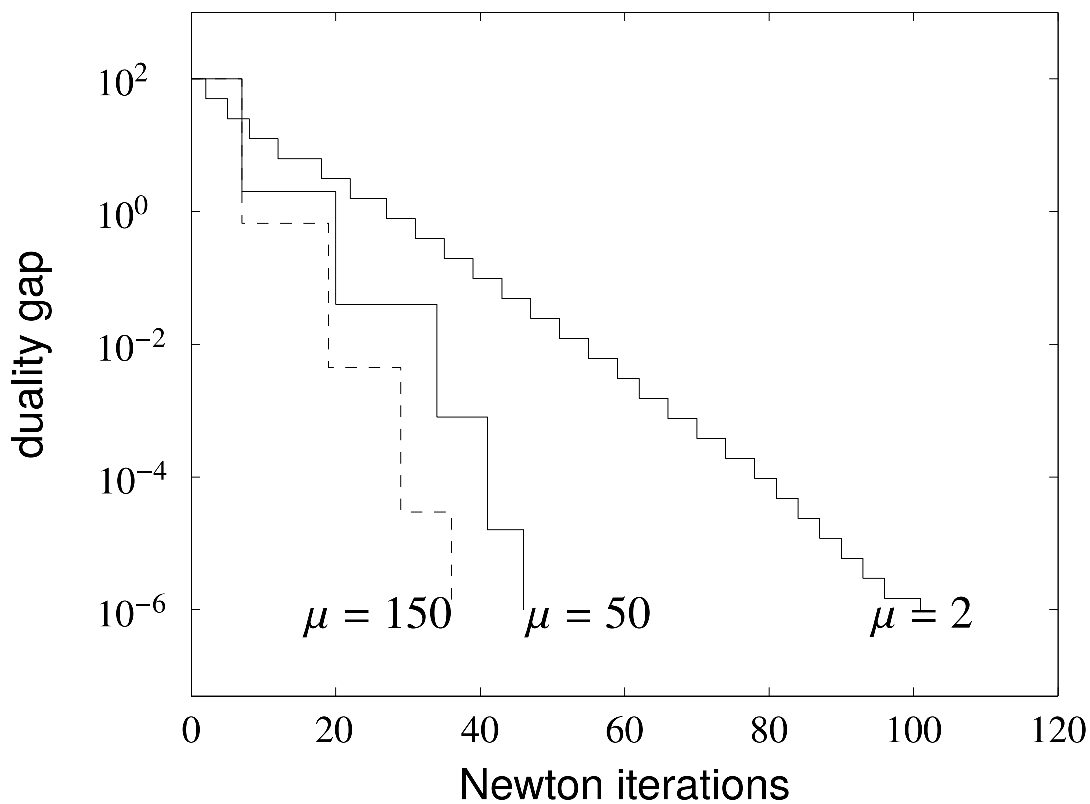


- starts with x on central path ($t^{(0)} = 1$, duality gap 100)
- terminates when $t = 10^8$ (gap 10^{-6})
- centering uses Newton's method with backtracking
- total number of Newton iterations not very sensitive for $\mu \geq 10$

Example: geometric program

GP with $m = 100$ inequalities and $n = 50$ variables

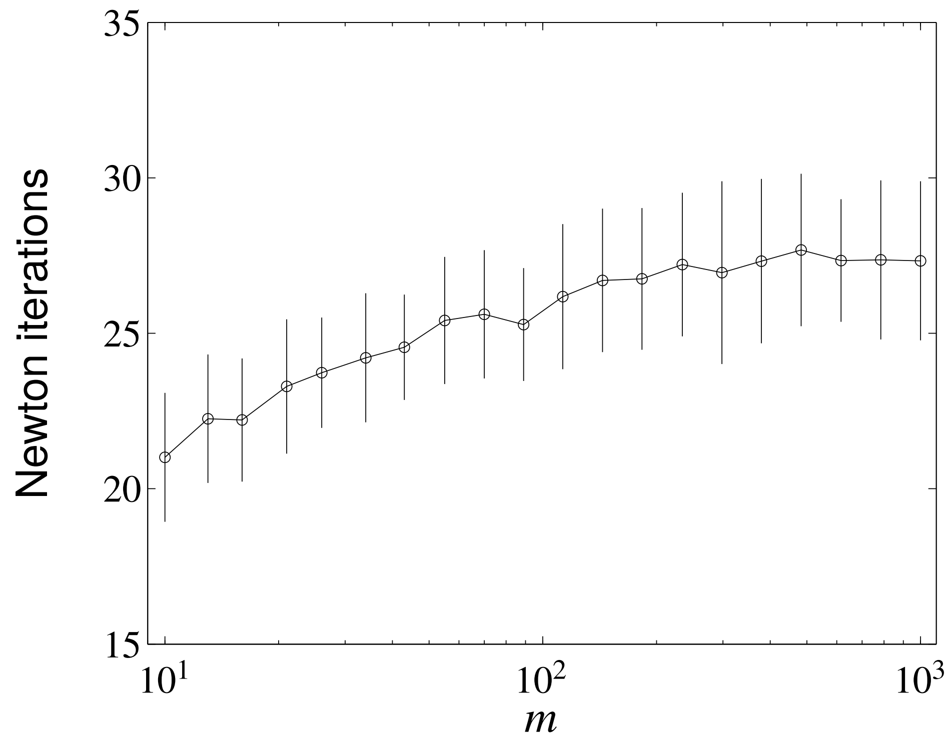
$$
\begin{array}{ll}
\text{minimize} & \log\left(\sum_{k=1}^{5} \exp(a_{0k}^T x + b_{0k})\right) \\
\text{subject to} & \log\left(\sum_{k=1}^{5} \exp(a_{ik}^T x + b_{ik})\right) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$



Example: family of standard LPs

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

- $A \in \mathbf{R}^{m \times 2m}$ with $m = 10, \ldots, 1000$
- for each m, solve 100 randomly generated instances



number of iterations grows very slowly as m ranges over a $100 : 1$ ratio

Feasibility and phase I methods

Phase I: computes a strictly feasible starting point, *i.e.*, x that satisfies

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Ax = b \tag{2}$$

Basic phase I method

$$\begin{array}{ll} \text{minimize (over } x, s) & s \\ \text{subject to} & f_i(x) \leq s, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{3}$$

- problem (3) is strictly feasible: take any x, s that satisfies

$$x \in \mathbf{dom}\, f_i, \quad i = 1, \ldots, m, \qquad Ax = b, \qquad s > \max_i f_i(x)$$

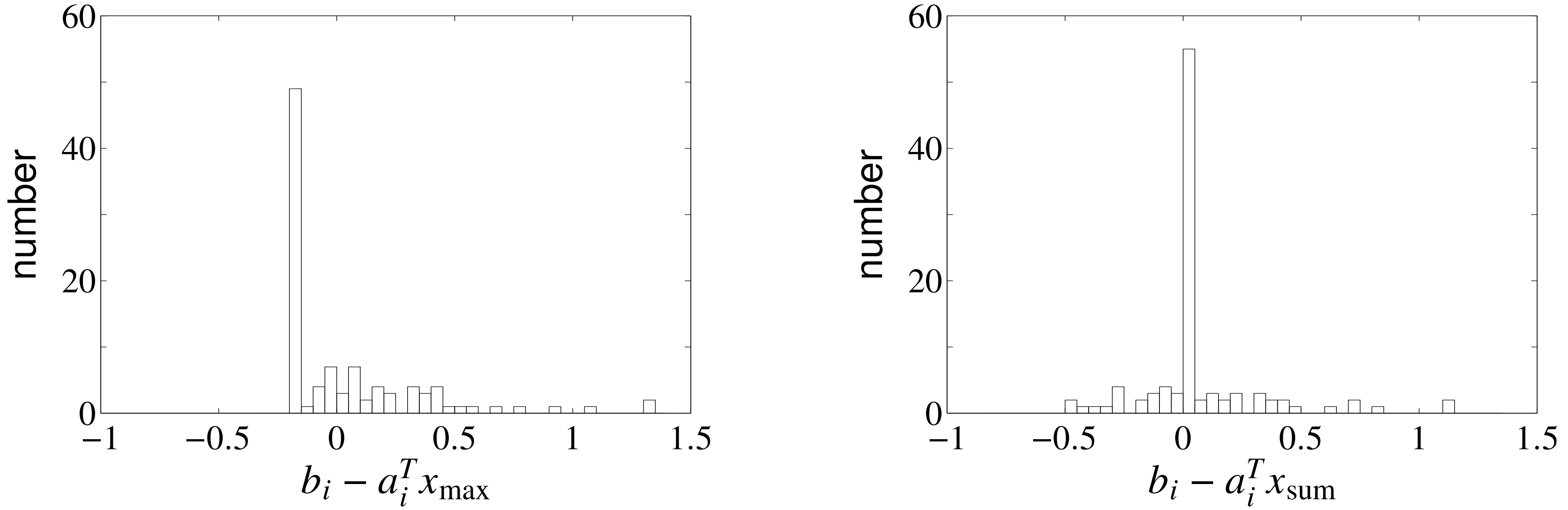
- if x, s are feasible for (3) with $s < 0$, then x is strictly feasible for (2)
- if optimal value $\bar{p}^\star$ of (3) is positive, then problem (2) is infeasible
- if $\bar{p}^\star = 0$ and attained, then problem (2) is feasible (but not strictly)
- if $\bar{p}^\star = 0$ and not attained, then problem (2) is infeasible

Sum of infeasibilities phase I method

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & s \succeq 0, \quad f_i(x) \leq s_i, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

for infeasible problem, will find x that satisfies many more inequalities than (3)

Example (infeasible set of 100 linear inequalities in 50 variables)



- left: basic phase I solution; satisfies 39 inequalities
- right: sum of infeasibilities phase I solution; satisfies 79 inequalities

Complexity analysis via self-concordance

same assumptions as on page 11.2, plus:

- sublevel sets (of f_0, on the feasible set) are bounded
- $tf_0 + \phi$ is self-concordant with closed sublevel sets

- second condition holds for LP, QP, QCQP
- may require reformulating the problem, *e.g.*,

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\
\text{subject to} & Fx \preceq g
\end{array}
\quad \longrightarrow \quad
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\
\text{subject to} & Fx \preceq g, \quad x \succeq 0
\end{array}
$$

- assumptions are needed for complexity analysis, not to run the barrier method

Newton iterations per centering step

bound on effort of computing $x^+ = x^\star(\mu t)$ starting at $x = x^\star(t)$:

$$\#\text{Newton iterations} \leq \frac{\mu t f_0(x) + \phi(x) - \mu t f_0(x^+) - \phi(x^+)}{\gamma} + c \qquad (4)$$

- γ, c are constants (depend only on algorithm parameters); see page 9.33
- upper bound on first term follows from duality:

$$\begin{aligned}
&\mu t f_0(x) + \phi(x) - \mu t f_0(x^+) - \phi(x^+) \\
&\quad = \quad \mu t f_0(x) - \mu t f_0(x^+) + \sum_{i=1}^{m} \log(-\mu t \lambda_i f_i(x^+)) - m \log \mu \\
&\quad \leq \quad \mu t f_0(x) - \mu t f_0(x^+) - \mu t \sum_{i=1}^{m} \lambda_i f_i(x^+) - m - m \log \mu \\
&\quad \leq \quad \mu t f_0(x) - \mu t g(\lambda, \nu) - m - m \log \mu \\
&\quad = \quad m(\mu - 1 - \log \mu)
\end{aligned}$$

where $\lambda_i = \lambda_i^\star(t) = -1/(t f_i(x^\star(t)))$

Total number of Newton iterations

- we exclude first centering step on page 11.11, assume we start at $x^\star(t^{(0)})$
- bound on Newton iterations is number of outer iterations times (4)

$$\#\text{Newton iterations} \leq N = \left\lceil \frac{\log(m/(t^{(0)}\epsilon))}{\log \mu} \right\rceil \left(\frac{m(\mu - 1 - \log \mu)}{\gamma} + c \right)$$

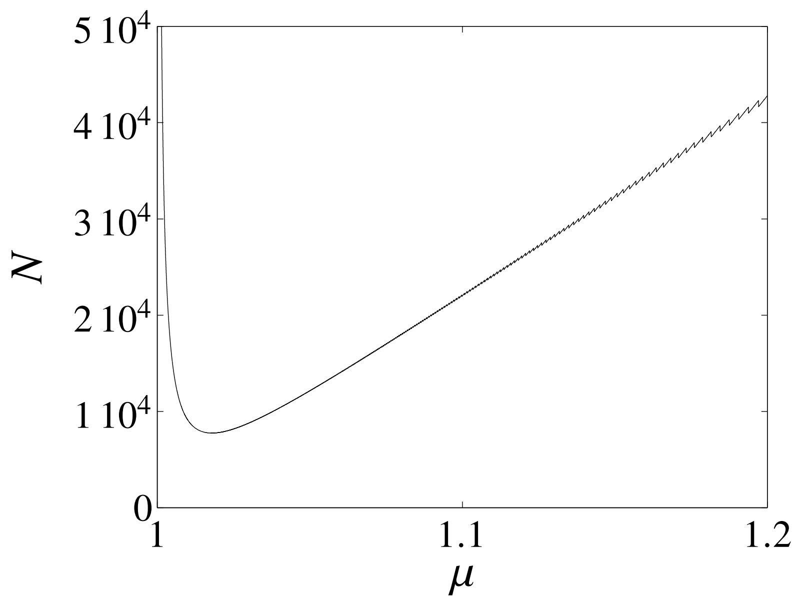


figure shows N for typical values of γ, c,

$$m = 100, \qquad \frac{m}{t^{(0)}\epsilon} = 10^5$$

- confirms trade-off in choice of μ
- in practice, #iterations is in the tens and not very sensitive for $\mu \geq 10$

Polynomial-time complexity of barrier method

- for $\mu = 1 + 1/\sqrt{m}$:

$$N = O\left(\sqrt{m}\log\left(\frac{m/t^{(0)}}{\epsilon}\right)\right)$$

- number of Newton iterations for fixed gap reduction is $O(\sqrt{m})$
- multiply with cost of one Newton iteration to get bound on number of flops
- this choice of μ optimizes worst-case complexity
- in practice we choose μ fixed ($\mu = 10, \ldots, 20$)

Second-order cone programming

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

- constraint functions are not differentiable
- barrier method for second-order cone programming uses barrier function

$$\begin{aligned} \phi(x) &= -\sum_{i=1}^m \log((c_i^T x + d_i)^2 - \|A_i x + b_i\|_2^2) \\ &= -\sum_{i=1}^m \log(c_i^T x + d_i) - \sum_{i=1}^m \log(c_i^T x + d_i - \frac{\|A_i x + b_i\|_2^2}{c_i^T x + d_i}) \end{aligned}$$

- equivalent to standard barrier method for reformulation with $2m$ inequalities

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \dfrac{\|A_i x + b_i\|_2^2}{c_i^T x + d_i} \le c_i^T x + d_i, \quad i = 1, \ldots, m \\ & c_i^T x + d_i \ge 0, \quad i = 1, \ldots, m \end{array}$$

Semidefinite programming

Primal and dual SDP (with $F_1, \ldots, F_n, G \in \mathbf{S}^m$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \sum_{i=1}^n x_i F_i \preceq G \end{array} \qquad \begin{array}{ll} \text{maximize} & -\operatorname{tr}(GZ) \\ \text{subject to} & \operatorname{tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots, n \\ & Z \succeq 0 \end{array}$$

Logarithmic barrier

$$\phi(x) = -\log\det F(x), \qquad \text{where } F(x) = G - \sum_{i=1}^n x_i F_i$$

- a convex differentiable function, with domain $\{x \mid F(x) \succ 0\}$
- gradient and Hessian are

$$\nabla\phi(x)_i = \operatorname{tr}(F_i F(x)^{-1}), \qquad \nabla^2\phi(x)_{ij} = \operatorname{tr}(F_i(F(x)^{-1} F_j F(x)^{-1}),$$

for $i, j = 1, \ldots, n$

Central path

points on central path $x^\star(t)$ for $t > 0$ are minimizers of $tc^T x + \phi(x)$

- optimality condition for centering problem:

$$0 = tc_i + \nabla\phi(x)_i = tc_i + \mathrm{tr}(F_i F(x)^{-1}), \quad i = 1, \ldots, n$$

- dual point on central path:

$$Z^\star(t) = \frac{1}{t} F(x^\star(t))^{-1}$$

- corresponding duality gap:

$$\begin{aligned} c^T x^\star(t) + \mathrm{tr}(GZ^\star(t)) &= \mathrm{tr}((-\sum_{i=1}^n x_i^\star(t) F_i + G) Z^\star(t)) \\ &= \mathrm{tr}(F(x^\star(t) Z^\star(t)) \\ &= m/t \end{aligned}$$

Barrier method for semidefinite programming

given: strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$

repeat

1. *centering step:* compute $x^\star(t)$ by minimizing $tc^T x + \phi(x)$
2. *update:* $x := x^\star(t)$
3. *stopping criterion:* quit if $m/t < \epsilon$
4. *increase t:* $t := \mu t$

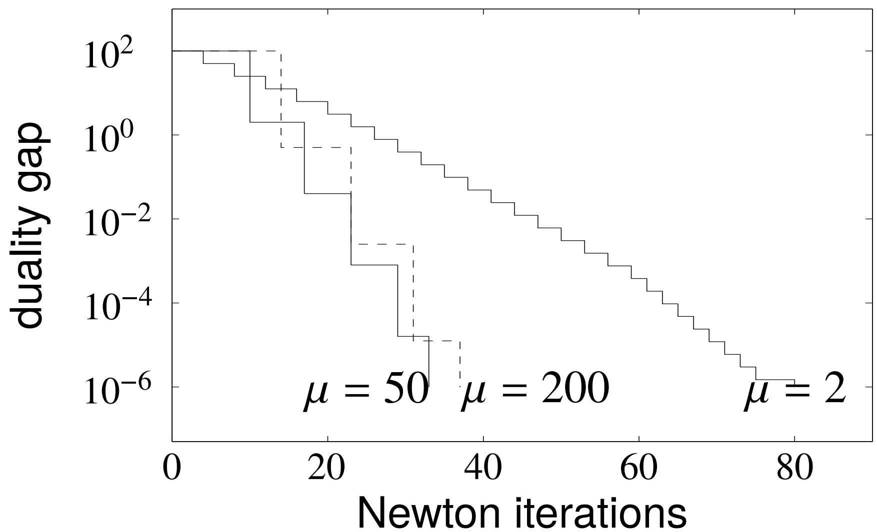
- number of outer iterations:

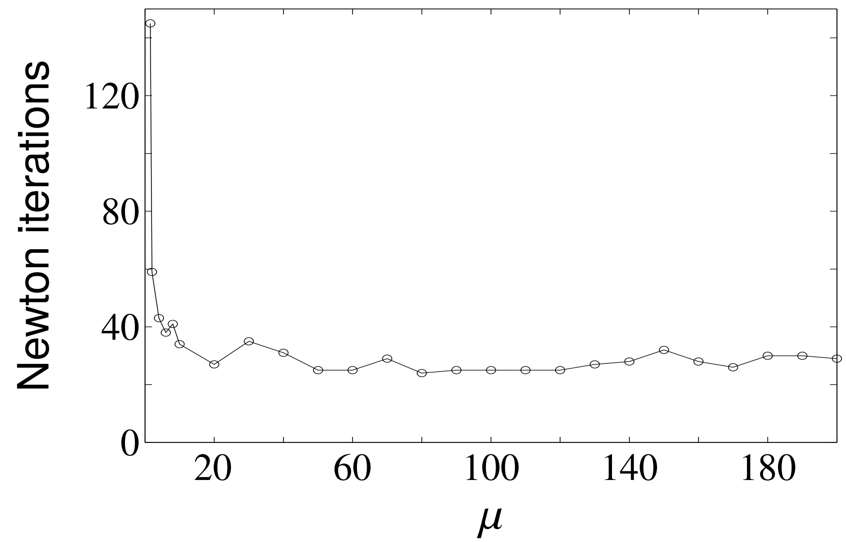
$$\left\lceil \frac{\log(m/(\epsilon t^{(0)}))}{\log \mu} \right\rceil$$

- complexity analysis via self-concordance also applies to SDP

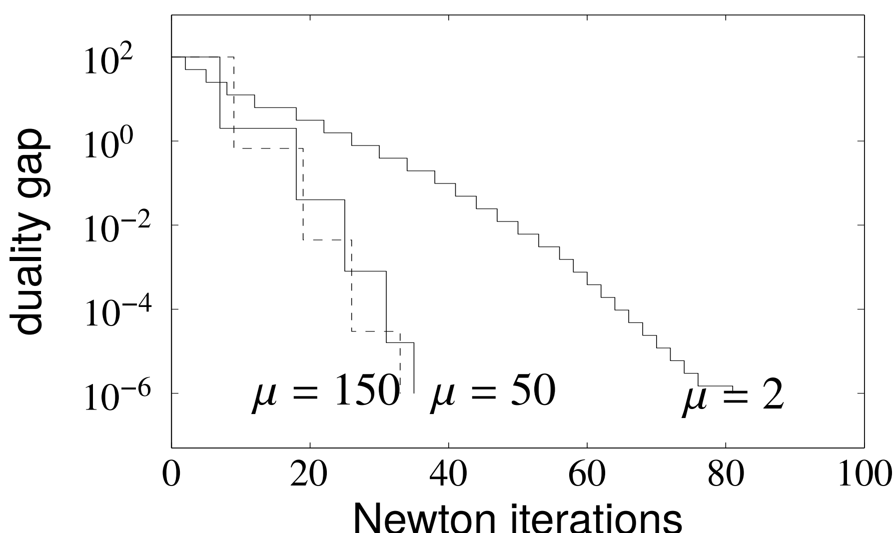
Examples

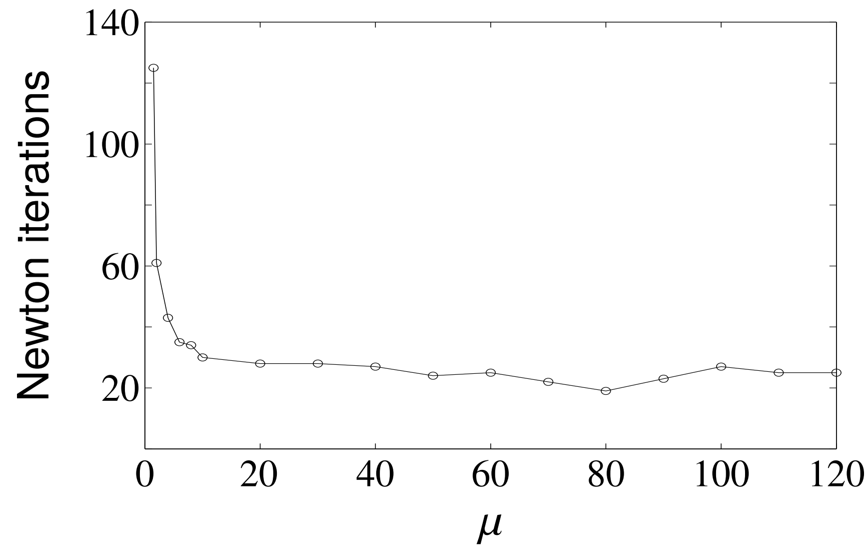
Second-order cone program (50 variables, 50 SOC constraints in $\mathbf{R}^6$





Semidefinite program (100 variables, constraint in $\mathbf{S}^{100}$)

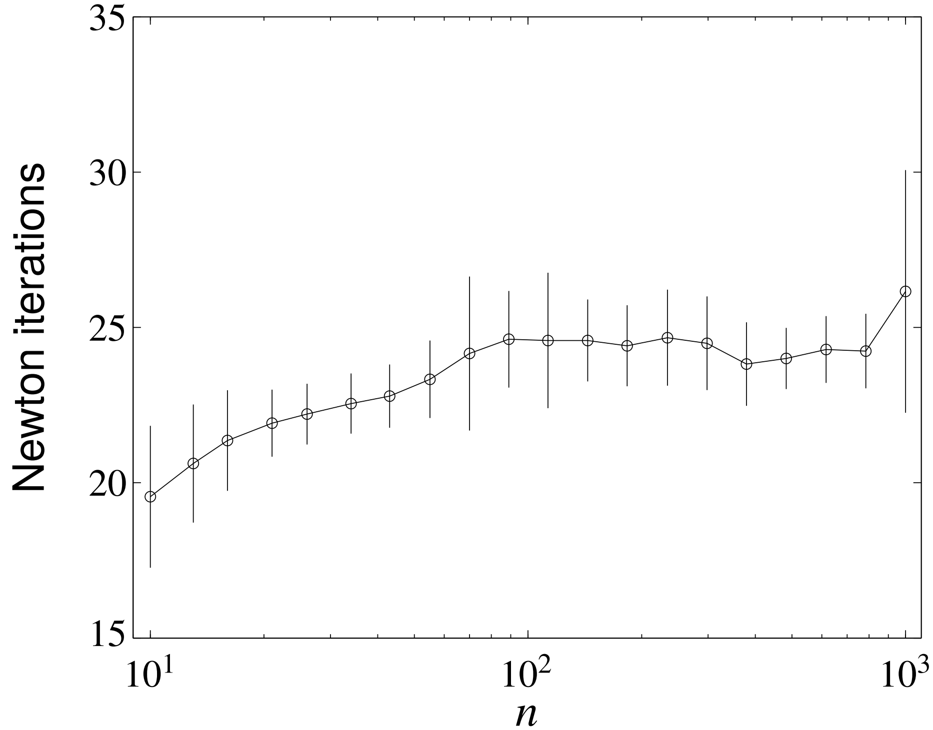




Family of SDPs ($A \in \mathbf{S}^n$, $x \in \mathbf{R}^n$)

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x \\ \text{subject to} & A + \mathbf{diag}(x) \succeq 0 \end{array}$$

$n = 10, \ldots, 1000$, for each n solve 100 randomly generated instances



Primal-dual interior-point methods

more efficient than barrier method when high accuracy is needed

- update primal and dual variables at each iteration
- no distinction between inner and outer iterations
- often exhibit superlinear asymptotic convergence
- steps can be interpreted as Newton iterates for modified KKT conditions
- can start at infeasible points
- cost per iteration same as barrier method