3. Convex functions

- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
Definition

\( f : \mathbb{R}^n \to \mathbb{R} \) is convex if \( \text{dom } f \) is a convex set and

\[
f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)
\]

for all \( x, y \in \text{dom } f \), \( 0 \leq \theta \leq 1 \)

- \( f \) is concave if \(-f\) is convex
- \( f \) is strictly convex if \( \text{dom } f \) is convex and

\[
f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)
\]

for \( x, y \in \text{dom } f \), \( x \neq y \), \( 0 < \theta < 1 \)
Examples on $\mathbb{R}$

**Convex**

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- exponential: $e^{ax}$, for any $a \in \mathbb{R}$
- powers: $x^\alpha$ on $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbb{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbb{R}_{++}$

**Concave**

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- powers: $x^\alpha$ on $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbb{R}_{++}$
Examples on $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

Examples on $\mathbb{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

Examples on $\mathbb{R}^{m \times n}$ ($m \times n$ matrices)

- affine function
  \[
  f(X) = \text{tr}(A^T X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b
  \]
- 2-norm (spectral norm): maximum singular value
  \[
  f(X) = \|X\|_2 = \sigma_{\text{max}}(X) = (\lambda_{\text{max}}(X^T X))^{1/2}
  \]
Restriction of a convex function to a line

\[ f : \mathbb{R}^n \to \mathbb{R} \] is convex if and only if the function \( g : \mathbb{R} \to \mathbb{R} \),

\[ g(t) = f(x + tv), \quad \text{dom } g = \{ t \mid x + tv \in \text{dom } f \} \]

is convex (in \( t \)) for any \( x \in \text{dom } f, v \in \mathbb{R}^n \)

can check convexity of \( f \) by checking convexity of functions of one variable

Example: \( f : \mathbb{S}^n \to \mathbb{R} \) with \( f(X) = \log \det X, \) \( \text{dom } f = \mathbb{S}_+^n \)

\[
g(t) = \log \det(X + tV) = \log \det X + \log \det(I + tX^{-1/2}VX^{-1/2})
\]

\[
= \log \det X + \sum_{i=1}^{n} \log(1 + t\lambda_i)
\]

where \( \lambda_i \) are the eigenvalues of \( X^{-1/2}VX^{-1/2} \)

\( g \) is concave in \( t \) (for any choice of \( X > 0, V \)); hence \( f \) is concave
Extended-value extension

extended-value extension \( \tilde{f} \) of \( f \) is

\[
\tilde{f}(x) = f(x), \quad x \in \text{dom} \, f, \quad \tilde{f}(x) = \infty, \quad x \notin \text{dom} \, f
\]

often simplifies notation; for example, the condition

\[
0 \leq \theta \leq 1 \implies \tilde{f}(\theta x + (1 - \theta)y) \leq \theta \tilde{f}(x) + (1 - \theta)\tilde{f}(y)
\]

(as an inequality in \( \mathbb{R} \cup \{\infty\} \)), means the same as the two conditions

- \( \text{dom} \, f \) is convex
- for \( x, y \in \text{dom} \, f \),

\[
0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)
\]
First-order condition

\( f \) is differentiable if \( \text{dom} \ f \) is open and the gradient

\[
\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)
\]

exists at each \( x \in \text{dom} \ f \)

**First-order condition:** differentiable \( f \) with convex domain is convex iff

\[
f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all} \ x, y \in \text{dom} \ f
\]

first-order approximation of \( f \) is global underestimator
Second-order conditions

\( f \) is **twice differentiable** if \( \text{dom } f \) is open and the Hessian \( \nabla^2 f(x) \in S^n \),

\[
\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,
\]

exists at each \( x \in \text{dom } f \)

**Second-order conditions:** for twice differentiable \( f \) with convex domain

- \( f \) is convex if and only if

\[
\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \text{dom } f
\]

- if \( \nabla^2 f(x) > 0 \) for all \( x \in \text{dom } f \), then \( f \) is strictly convex
Examples

Quadratic function: $f(x) = (1/2)x^TPx + q^Tx + r$ (with $P \in S^n$)

$$\nabla f(x) = Px + q, \quad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

Least squares objective: $f(x) = \|Ax - b\|_2^2$

$$\nabla f(x) = 2A^T(Ax - b), \quad \nabla^2 f(x) = 2A^TA$$

convex (for any $A$)

Quadratic-over-linear function: $f(x, y) = x^2/y$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \left[ \begin{array}{c} y \\ -x \end{array} \right] \left[ \begin{array}{c} y \\ -x \end{array} \right]^T \succeq 0$$

convex for $y > 0$
Examples

Log-sum-exp function: \( f(x) = \log \sum_{k=1}^{n} \exp x_k \) is convex

\[
\nabla^2 f(x) = \frac{1}{\mathbf{1}^T z} \text{diag}(z) - \frac{1}{(\mathbf{1}^T z)^2} z z^T \quad \text{with } z_k = \exp x_k
\]

To show \( \nabla^2 f(x) \geq 0 \), we must verify that \( v^T \nabla^2 f(x) v \geq 0 \) for all \( v \):

\[
v^T \nabla^2 f(x) v = \frac{\left( \sum_{k=1}^{n} z_k v_k^2 \right) \left( \sum_{k=1}^{n} z_k \right) - \left( \sum_{k=1}^{n} v_k z_k \right)^2}{\left( \sum_{k=1}^{n} z_k \right)^2} \geq 0
\]

since \( (\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k) \) (from Cauchy–Schwarz inequality)

Geometric mean: \( f(x) = \left( \prod_{k=1}^{n} x_k \right)^{1/n} \) on \( \mathbb{R}^n_{++} \) is concave

(similar proof as for log-sum-exp)
Epigraph and sublevel set

$\alpha$-sublevel set of $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$C_\alpha = \{x \in \text{dom } f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

**Epigraph** of $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\text{epi } f = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \text{dom } f, \ f(x) \leq t\}$$

$f$ is convex if and only if $\text{epi } f$ is a convex set
Jensen’s inequality

**Basic inequality:** if $f$ is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y)$$

**Extension:** if $f$ is convex, then

$$f(\mathbf{E} z) \leq \mathbf{E} f(z)$$

for any random variable $z$

basic inequality is special case with discrete distribution

$$\text{prob}(z = x) = \theta, \quad \text{prob}(z = y) = 1 - \theta$$
Operations that preserve convexity

methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that $f$ is obtained from simple convex functions by operations that preserve convexity
   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective
Positive weighted sum and composition with affine function

**Nonnegative multiple:** $\alpha f$ is convex if $f$ is convex, $\alpha \geq 0$

**Sum:** $f_1 + f_2$ convex if $f_1, f_2$ convex (extends to infinite sums, integrals)

**Composition with affine function:** $f(Ax + b)$ is convex if $f$ is convex

**Examples**
- logarithmic barrier for linear inequalities
  \[
  f(x) = - \sum_{i=1}^{m} \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i, \ i = 1, \ldots, m\}
  \]
- (any) norm of affine function: $f(x) = ||Ax + b||$
Pointwise maximum

if $f_1, \ldots, f_m$ are convex, then

$$f(x) = \max \{f_1(x), \ldots, f_m(x)\}$$

is convex

Examples

• piecewise-linear function: $f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$ is convex

• sum of $r$ largest components of $x \in \mathbb{R}^n$:

$$f(x) = x[1] + x[2] + \cdots + x[r]$$

is convex ($x[i]$ is $i$th largest component of $x$)

proof:

$$f(x) = \max \{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$
Pointwise supremum

if \( f(x, y) \) is convex in \( x \) for each \( y \in \mathcal{A} \), then

\[
g(x) = \sup_{y \in \mathcal{A}} f(x, y)
\]

is convex

Examples

- **support function** of a set: \( S_C(x) = \sup_{y \in C} y^T x \) is convex for any set \( C \)

- distance to farthest point in a set \( C \):

\[
f(x) = \sup_{y \in C} \|x - y\|
\]

- maximum eigenvalue of symmetric matrix: for \( X \in \mathcal{S}^n \),

\[
\lambda_{\text{max}}(X) = \sup_{\|y\|_2 = 1} y^T X y
\]
Composition with scalar functions

composition of \( g : \mathbb{R}^n \to \mathbb{R} \) and \( h : \mathbb{R} \to \mathbb{R} \):

\[
f(x) = h(g(x))
\]

\( f \) is convex if \( g \) convex, \( h \) convex, \( \tilde{h} \) nondecreasing
\( g \) concave, \( h \) convex, \( \tilde{h} \) nonincreasing

• proof (for \( n = 1 \), differentiable \( g, h \))

\[
f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)
\]

• note: monotonicity must hold for extended-value extension \( \tilde{h} \)

Examples

• \( \exp g(x) \) is convex if \( g \) is convex

• \( 1/g(x) \) is convex if \( g \) is concave and positive
Vector composition

composition of \( g : \mathbb{R}^n \to \mathbb{R}^k \) and \( h : \mathbb{R}^k \to \mathbb{R} \):

\[
f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))
\]

\( f \) is convex if \( g_i \) convex, \( h \) convex, \( \tilde{h} \) nondecreasing in each argument
\( g_i \) concave, \( h \) convex, \( \tilde{h} \) nonincreasing in each argument

proof (for \( n = 1 \), differentiable \( g, h \))

\[
f''(x) = (g'(x))^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)
\]

Examples

- \( \sum_{i=1}^{m} \log g_i(x) \) is concave if \( g_i \) are concave and positive
- \( \log \sum_{i=1}^{m} \exp g_i(x) \) is convex if \( g_i \) are convex
Minimization

if \( f(x, y) \) is convex in \((x, y)\) and \( C \) is a convex set, then

\[
g(x) = \inf_{y \in C} f(x, y)
\]

is convex

Examples

- \( f(x, y) = x^T A x + 2x^T B y + y^T C y \) with

\[
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix} \succeq 0, \quad C > 0
\]

minimizing over \( y \) gives

\[
g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T)x
\]

\( g \) is convex, hence Schur complement \( A - BC^{-1}B^T \succeq 0 \)

- distance to a set: \( d(x, S) = \inf_{y \in S} \|x - y\| \) is convex if \( S \) is convex
the **perspective** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$g(x, t) = tf(x/t), \quad \text{dom } g = \{(x, t) \mid x/t \in \text{dom } f, \ t > 0\}$$

$g$ is convex if $f$ is convex

**Examples**

- $f(x) = x^T x$ is convex; hence $g(x, t) = x^T x/t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy
  $$g(x, t) = t \log t - t \log x$$
  is convex on $\mathbb{R}^2_{++}$
- if $f$ is convex, then
  $$g(x) = (c^T x + d)f\left(\frac{(Ax + b)}{(c^T x + d)}\right)$$
  is convex on \{x \mid c^T x + d > 0, \ (Ax + b)/(c^T x + d) \in \text{dom } f\}
The conjugate function

The conjugate of a function \( f \) is

\[
f^*(y) = \sup_{x \in \text{dom } f} (y^T x - f(x))
\]

- \( f^* \) is convex (even if \( f \) is not)
- will be useful in chapter 5
Examples

- negative logarithm $f(x) = -\log x$

  $$f^*(y) = \sup_{x>0} (xy + \log x)$$

  $$= \begin{cases} 
  -1 - \log(-y) & y < 0 \\
  \infty & \text{otherwise}
  \end{cases}$$

- strictly convex quadratic $f(x) = (1/2)x^T Q x$ with $Q \in S^n_{++}$

  $$f^*(y) = \sup_x (y^T x - (1/2)x^T Q x)$$

  $$= \frac{1}{2} y^T Q^{-1} y$$
Quasiconvex functions

\( f : \mathbb{R}^n \to \mathbb{R} \) is quasiconvex if \( \text{dom } f \) is convex and the sublevel sets

\[
S_\alpha = \{ x \in \text{dom } f \mid f(x) \leq \alpha \}
\]

are convex for all \( \alpha \)

- \( f \) is quasiconcave if \( -f \) is quasiconvex
- \( f \) is quasilinear if it is quasiconvex and quasiconcave
Examples

- \( \sqrt{|x|} \) is quasiconvex on \( \mathbb{R} \)
- ceil\( (x) = \inf\{z \in \mathbb{Z} \mid z \geq x\} \) is quasilinear
- \( \log x \) is quasilinear on \( \mathbb{R}_{++} \)
- \( f(x_1, x_2) = x_1 x_2 \) is quasiconcave on \( \mathbb{R}_+^2 \)
- linear-fractional function
  \[
  f(x) = \frac{a^T x + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}
  \]
  is quasilinear
- distance ratio
  \[
  f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \quad \text{dom } f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}
  \]
  is quasiconvex
Internal rate of return

- cash flow \( x = (x_0, \ldots, x_n) \); \( x_i \) is payment in period \( i \) (to us if \( x_i > 0 \))

- we assume \( x_0 < 0 \) and \( x_0 + x_1 + \cdots + x_n > 0 \)

- present value of cash flow \( x \), for interest rate \( r \):

\[
PV(x, r) = \sum_{i=0}^{n} (1 + r)^{-i} x_i
\]

- internal rate of return is smallest interest rate for which \( PV(x, r) = 0 \):

\[
IRR(x) = \inf\{r \geq 0 \mid PV(x, r) = 0\}
\]

IRR is quasiconcave: superlevel set is intersection of open halfspaces

\[
IRR(x) \geq R \iff \sum_{i=0}^{n} (1 + r)^{-i} x_i > 0 \text{ for } 0 \leq r < R
\]
Properties

**Modified Jensen inequality:** for quasiconvex $f$

$$0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\}$$

**First-order condition:** differentiable $f$ with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \implies \nabla f(x)^T(y - x) \leq 0$$

**Sums:** sums of quasiconvex functions are not necessarily quasiconvex
Log-concave and log-convex functions

A positive function $f$ is log-concave if $\log f$ is concave:

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta} \text{ for } 0 \leq \theta \leq 1$$

$f$ is log-convex if $\log f$ is convex

- Powers: $x^a$ on $\mathbb{R}_{++}$ is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- Many common probability densities are log-concave, e.g., normal:
  $$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$

- Cumulative Gaussian distribution function $\Phi$ is log-concave
  $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$
Properties of log-concave functions

• twice differentiable $f$ with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \leq \nabla f(x)\nabla f(x)^T \quad \text{for all } x \in \text{dom} \, f$$

• product of log-concave functions is log-concave

• sum of log-concave functions is not always log-concave

• integration: if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is log-concave, then

$$g(x) = \int f(x, y) \, dy$$

is log-concave (not easy to show)
Consequences of integration property

- Convolution $f * g$ of log-concave functions $f, g$ is log-concave

  $$(f * g)(x) = \int f(x - y)g(y)dy$$

- If $C \subseteq \mathbb{R}^n$ convex and $y$ is a random variable with log-concave p.d.f. then

  $$f(x) = \text{prob}(x + y \in C)$$

  is log-concave

Proof: write $f(x)$ as integral of product of log-concave functions

  $$f(x) = \int g(x + y)p(y)dy, \quad g(u) = \begin{cases} 
  1 & u \in C \\
  0 & u \notin C, 
\end{cases}$$

  $p$ is p.d.f. of $y$
Example: yield function

\[ Y(x) = \text{prob}(x + w \in S) \]

- \( x \in \mathbb{R}^n \): nominal parameter values for product
- \( w \in \mathbb{R}^n \): random variations of parameters in manufactured product
- \( S \): set of acceptable values

If \( S \) is convex and \( w \) has a log-concave p.d.f., then

- \( Y \) is log-concave
- yield regions \( \{ x \mid Y(x) \geq \alpha \} \) are convex