3. Convex functions

- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
**Definition**

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\text{dom } f$ is a convex set and

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

for all $x, y \in \text{dom } f$, $0 \leq \theta \leq 1$

- $f$ is concave if $-f$ is convex
- $f$ is strictly convex if $\text{dom } f$ is convex and

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$$

for $x, y \in \text{dom } f$, $x \neq y$, $0 < \theta < 1$
Examples on $\mathbb{R}$

**Convex**

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- exponential: $e^{ax}$, for any $a \in \mathbb{R}$
- powers: $x^\alpha$ on $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbb{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbb{R}_{++}$

**Concave**

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- powers: $x^\alpha$ on $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbb{R}_{++}$
Examples on $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$

- Affine functions are convex and concave
- All norms are convex

Examples on $\mathbb{R}^n$

- Affine function $f(x) = a^T x + b$
- Norms: $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

Examples on $\mathbb{R}^{m \times n}$ ($m \times n$ matrices)

- Affine function
  \[ f(X) = \text{tr}(A^T X) + b = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} X_{ij} + b \]
- 2-norm (spectral norm): maximum singular value
  \[ f(X) = \|X\|_2 = \sigma_{\text{max}}(X) = (\lambda_{\text{max}}(X^T X))^{1/2} \]
Extended-value extension

extended-value extension \( \tilde{f} \) of \( f \) is

\[
\tilde{f}(x) = f(x), \quad x \in \text{dom } f, \quad \tilde{f}(x) = \infty, \quad x \not\in \text{dom } f
\]

often simplifies notation; for example, the condition

\[
0 \leq \theta \leq 1 \implies \tilde{f}(\theta x + (1 - \theta)y) \leq \theta \tilde{f}(x) + (1 - \theta)\tilde{f}(y)
\]

(as an inequality in \( \mathbb{R} \cup \{\infty\} \)), means the same as the two conditions

- \( \text{dom } f \) is convex
- for \( x, y \in \text{dom } f \),

\[
0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)
\]
Restriction of a convex function to a line

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = f(x + tv), \quad \text{dom } g = \{t \mid x + tv \in \text{dom } f\}$$

is convex (in $t$) for any $x \in \text{dom } f$, $v \in \mathbb{R}^n$

can check convexity of $f$ by checking convexity of functions of one variable

Example: $f : S^n \to \mathbb{R}$ with $f(X) = \log \det X$, dom $f = S^n_{++}$

$$g(t) = \log \det(X + tV) = \log \det X + \log \det(I + tX^{-1/2}VVX^{-1/2})$$

$$= \log \det X + \sum_{i=1}^n \log(1 + t\lambda_i)$$

where $\lambda_i$ are the eigenvalues of $X^{-1/2}VVX^{-1/2}$

$g$ is concave in $t$ (for any choice of $X > 0$, $V$); hence $f$ is concave
First-order condition

$f$ is **differentiable** if $\text{dom } f$ is open and the gradient

$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \text{dom } f$

**First-order condition:** differentiable $f$ with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \text{dom } f$$

first-order approximation of $f$ is global underestimator
Second-order conditions

\( f \) is **twice differentiable** if \( \text{dom} \ f \) is open and the Hessian \( \nabla^2 f(x) \in S^n \),

\[
\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,
\]

exists at each \( x \in \text{dom} \ f \)

**Second-order conditions:** for twice differentiable \( f \) with convex domain

- \( f \) is convex if and only if

\[
\nabla^2 f(x) \succeq 0 \quad \text{for all} \ x \in \text{dom} \ f
\]

- if \( \nabla^2 f(x) > 0 \) for all \( x \in \text{dom} \ f \), then \( f \) is strictly convex
Examples

**Quadratic function:** \( f(x) = (1/2)x^T P x + q^T x + r \) (with \( P \in S^n \))

\[
\nabla f(x) = Px + q, \quad \nabla^2 f(x) = P
\]

convex if \( P \succeq 0 \)

**Least squares objective:** \( f(x) = \|Ax - b\|_2^2 \)

\[
\nabla f(x) = 2A^T (Ax - b), \quad \nabla^2 f(x) = 2A^T A
\]

convex (for any \( A \))

**Quadratic-over-linear function:** \( f(x, y) = x^2 / y \)

\[
\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y & y \end{bmatrix} \begin{bmatrix} -y & -x \end{bmatrix}^T \succeq 0
\]

\( \text{dom } f = \{(x, y) \mid y > 0\} \)
Examples

**Log-sum-exp function:** $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{1^T z} \text{diag}(z) - \frac{1}{(1^T z)^2} zz^T \quad \text{with } z_k = \exp x_k$$

to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^T \nabla^2 f(x) v \geq 0$ for all $v$:

$$v^T \nabla^2 f(x) v = \frac{\left( \sum_{k=1}^{n} z_k v_k^2 \right) \left( \sum_{k=1}^{n} z_k \right) - \left( \sum_{k=1}^{n} v_k z_k \right)^2}{\left( \sum_{k=1}^{n} z_k \right)^2} \geq 0$$

since $(\sum_{k} v_k z_k)^2 \leq (\sum_{k} z_k v_k^2)(\sum_{k} z_k)$ (from Cauchy–Schwarz inequality)

**Geometric mean:** $f(x) = (\prod_{k=1}^{n} x_k)^{1/n}$ on $\mathbb{R}_+^n$ is concave

(similar proof as for log-sum-exp)
**Epigraph and sublevel set**

**α-sublevel set** of $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$C_\alpha = \{x \in \text{dom } f \mid f(x) \leq \alpha\}$$

Sublevel sets of convex functions are convex (converse is false)

**Epigraph** of $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\text{epi } f = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \text{dom } f, \ f(x) \leq t\}$$

$f$ is convex if and only if $\text{epi } f$ is a convex set
Jensen’s inequality

**Basic inequality:** if $f$ is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

**Extension:** if $f$ is convex, then

$$f(\mathbb{E} z) \leq \mathbb{E} f(z)$$

for any random variable $z$

basic inequality is special case with discrete distribution

$$\text{prob}(z = x) = \theta, \quad \text{prob}(z = y) = 1 - \theta$$
Operations that preserve convexity

methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that $f$ is obtained from simple convex functions by operations that preserve convexity
   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective
Positive weighted sum and composition with affine function

Nonnegative multiple: $\alpha f$ is convex if $f$ is convex, $\alpha \geq 0$

Sum: $f_1 + f_2$ convex if $f_1, f_2$ convex (extends to infinite sums, integrals)

Composition with affine function: $f(Ax + b)$ is convex if $f$ is convex

Examples

- logarithmic barrier for linear inequalities

  $$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i, \ i = 1, \ldots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$
Pointwise maximum

If $f_1, \ldots, f_m$ are convex, then the function

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is convex

Examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of $r$ largest components of $x \in \mathbb{R}^n$:

$$f(x) = x[1] + x[2] + \cdots + x[r]$$

is convex ($x[i]$ is $i$th largest component of $x$)

proof: $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$
Pointwise supremum

if $f(x, y)$ is convex in $x$ for each $y \in \mathcal{A}$, then the function

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

Examples

• *support function* of a set: $S_C(x) = \sup_{y \in C} y^T x$ is convex for any set $C$
• distance to farthest point in a set $C$:

$$f(x) = \sup_{y \in C} \|x - y\|$$

• maximum eigenvalue of symmetric matrix: for $X \in \mathbb{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$
Proof.

• suppose $f(x, y)$ is a convex function of $x$, for any fixed $y \in \mathcal{A}$ (and $\mathcal{A} \neq \emptyset$)

• this means that for all $y \in \mathcal{A}$, $x_1, x_2$, and $\theta \in [0, 1]$,

\[
f(\theta x_1 + (1 - \theta)x_2, y) \leq \theta f(x_1, y) + (1 - \theta) f(x_2, y)
\]

• for simplicity, we use the extended-value convention (where $0 \cdot \infty = 0$)

convexity of $g$ follows from

\[
g(\theta x_1 + (1 - \theta)x_2) = \sup_{y \in \mathcal{A}} f(\theta x_1 + (1 - \theta)x_2, y)
\]
\[
\leq \sup_{y \in \mathcal{A}} (\theta f(x_1, y) + (1 - \theta) f(x_2, y))
\]
\[
\leq \theta \sup_{y \in \mathcal{A}} f(x_1, y) + (1 - \theta) \sup_{y \in \mathcal{A}} f(x_2, y)
\]
\[
= \theta g(x_1) + (1 - \theta) g(x_2)
\]
Partial minimization

if \( f(x, y) \) is convex in \( (x, y) \) and \( C \) is a convex set, then the function

\[
g(x) = \inf_{y \in C} f(x, y)
\]

is convex

Examples

- \( f(x, y) = x^T A x + 2x^T B y + y^T C y \) with

\[
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix} \succeq 0, \quad C > 0
\]

minimizing over \( y \) gives

\[
g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T)x
\]

\( g \) is convex, hence Schur complement \( A - BC^{-1}B^T \succeq 0 \)

- distance to a set: \( d(x, S) = \inf_{y \in S} \|x - y\| \) is convex if \( S \) is convex
Proof. 

• suppose \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is jointly convex in \((x, y)\)

• without loss of generality, we make \( C \) part of the domain of \( f \), i.e., redefine

\[
\text{dom } f := \text{dom } f \cap \{(x, y) \mid y \in C\}, \quad C := \mathbb{R}^m
\]

• convexity of \( f(x, y) \) jointly in \((x, y)\) means that for all \( x_1, y_1, x_2, y_2, \theta \in [0, 1] \),

\[
f(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_2) \leq \theta f(x_1, y_1) + (1 - \theta)f(x_2, y_2)
\]

• assume \( \inf_y f(x, y) > -\infty \) for all \( x \) (we don’t allow functions that take value \(-\infty\))

convexity of \( g \) follows from

\[
g(\theta x_1 + (1 - \theta)x_2) = \inf_y f(\theta x_1 + (1 - \theta)x_2, y) \\
= \inf_{y_1, y_2} f(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_2) \\
\leq \inf_{y_1, y_2} (\theta f(x_1, y_1) + (1 - \theta)f(x_2, y_2)) \\
\leq \theta \inf_{y_1} f(x_1, y_1) + (1 - \theta) \inf_{y_2} f(x_2, y_2) \\
= \theta g(x_1) + (1 - \theta)g(x_2)
\]
Summary of minimization/maximization rules

if we include the counterparts for concave functions, there are four rules

Maximization

\[ g(x) = \sup_{y \in C} f(x, y) \]

- \( g \) is convex if \( f \) is convex in \( x \) for fixed \( y \); \( C \) can be any set
- \( g \) is concave if \( f \) is jointly concave in \( (x, y) \) and \( C \) is a convex set

Minimization

\[ g(x) = \inf_{y \in C} f(x, y) \]

- \( g \) is convex if \( f \) is jointly convex in \( (x, y) \) and \( C \) is a convex set
- \( g \) is concave if \( f \) is concave in \( x \) for fixed \( y \); \( C \) can be any set
Composition with scalar functions

composition of $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$:

$$f(x) = h(g(x))$$

$f$ is convex if $h$ is convex and one of the following three cases holds

- $g$ is convex and $\tilde{h}$ nondecreasing
- $g$ is concave and $\tilde{h}$ nonincreasing
- $g$ is affine

- monotonicity properties of $h$ must hold for extended-value extension $\tilde{h}$
- quick proof (for $n = 1$, differentiable $g, h$)

$$f''''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

Examples

- $\exp g(x)$ is convex if $g$ is convex
- $1/g(x)$ is convex if $g$ is concave and positive
Proof (first composition rule)

• suppose $g$ is convex and $h$ is convex

• suppose $\tilde{h}$ is nondecreasing: this means that

$$y \leq x, \ x \in \text{dom } h \quad \implies \quad y \in \text{dom } h, \ h(y) \leq h(x)$$

consider convex combination of points $x_1, x_2 \in \text{dom } f$

• $x_1, x_2 \in \text{dom } f$ means that $x_1, x_2 \in \text{dom } g$ and $g(x_1), g(x_2) \in \text{dom } h$

• by convexity of $g$, the convex combination $\theta x_1 + (1 - \theta)x_2$ is in $\text{dom } g$ and

$$g(\theta x_1 + (1 - \theta)x_2) \leq \theta g(x_1) + (1 - \theta)g(x_2)$$

• by monotonicity of $\tilde{h}$ and convexity of $h$, $g(\theta x_1 + (1 - \theta)x_2) \in \text{dom } h$ and

$$h(g(\theta x_1 + (1 - \theta)x_2)) \leq h(\theta g(x_1) + (1 - \theta)g(x_2)) \leq \theta h(g(x_1)) + (1 - \theta)h(g(x_2))$$
Vector composition

composition of \( g : \mathbb{R}^n \rightarrow \mathbb{R}^k \) and \( h : \mathbb{R}^k \rightarrow \mathbb{R} \):

\[
f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))
\]

\( f \) is convex if \( h \) is convex and for each \( i \), one of the following three cases holds

- \( g_i \) is convex and \( \tilde{h} \) nondecreasing in its \( i \)th argument
- \( g_i \) is concave and \( \tilde{h} \) is nonincreasing in its \( i \)th argument
- \( g_i \) is affine

- \( \tilde{h} \) is extended-value extension of \( h \)
- quick proof (for \( n = 1 \), differentiable \( g, h \))

\[
f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)
\]

Examples

- \( \sum_{i=1}^{m} \log g_i(x) \) is concave if \( g_i \) are concave and positive
- \( \log \sum_{i=1}^{m} \exp g_i(x) \) is convex if \( g_i \) are convex
Proof (first composition rule)

- suppose $g_1, \ldots, g_k$ is convex and $h$ is convex
- Jensen’s inequality for $g_1, \ldots, g_k$ can be written as a vector inequality
  \[ g(\theta x_1 + (1 - \theta)x_2) \leq \theta g(x_1) + (1 - \theta)g(x_2) \]
- suppose $\tilde{h}$ is nondecreasing in each argument; this means that
  \[ y \leq x, \ x \in \text{dom } h \implies y \in \text{dom } h, \ h(y) \leq h(x) \]

consider a convex combination of points $x_1, x_2 \in \text{dom } f$

- $x_1, x_2 \in \text{dom } f$ means that $x_1, x_2 \in \text{dom } g$ and $g(x_1), g(x_2) \in \text{dom } h$
- by convexity of $g_1, \ldots, g_k$, the convex combination $\theta x_1 + (1 - \theta)x_2 \in \text{dom } g$ and
  \[ g(\theta x_1 + (1 - \theta)x_2) \leq \theta g(x_1) + (1 - \theta)g(x_2) \]

- hence, by monotonicity of $\tilde{h}$ and convexity of $h$, $g(\theta x_1 + (1 - \theta)x_2) \in \text{dom } h$ and
  \[ h(g(\theta x_1 + (1 - \theta)x_2)) \leq h(\theta g(x_1) + (1 - \theta)g(x_2)) \leq \theta h(g(x_1)) + (1 - \theta)h(g(x_2)) \]
the **perspective** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$g(x, t) = tf(x/t), \quad \text{dom } g = \{(x, t) \mid x/t \in \text{dom } f, \ t > 0\}$$

$g$ is convex if $f$ is convex

**Examples**

- $f(x) = x^Tx$ is convex; hence $g(x, t) = x^T x/t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy

$$g(x, t) = t \log t − t \log x$$

is convex on $\mathbb{R}_+^2$
- if $f$ is convex, then

$$g(x) = (c^T x + d) f \left( (Ax + b)/(c^T x + d) \right)$$

is convex on $\{x \mid c^T x + d > 0, \ (Ax + b)/(c^T x + d) \in \text{dom } f\}$
Proof.

- consider convex combination of two points \((x_1, t_1), (x_2, t_2) \in \text{dom } g\):
  \[
t_1 > 0, \quad x_1/t_1 \in \text{dom } f, \quad t_2 > 0, \quad x_2/t_2 \in \text{dom } f
  \]

- we verify Jensen’s inequality:
  \[
g(\theta x_1 + (1 - \theta)x_2, \theta t_1 + (1 - \theta)t_2)
  \]
  \[
  = (\theta t_1 + (1 - \theta)t_2) f\left(\frac{\theta x_1 + (1 - \theta)x_2}{\theta t_1 + (1 - \theta)t_2}\right)
  \]
  \[
  = (\theta t_1 + (1 - \theta)t_2) f\left(\frac{\theta t_1}{\theta t_1 + (1 - \theta)t_2}(x_1/t_1) + \frac{(1 - \theta)t_2}{\theta t_1 + (1 - \theta)t_2}(x_2/t_2)\right)
  \]
  \[
  \leq \theta t_1 f(x_1/t_1) + (1 - \theta)t_2 f(x_2/t_2)
  \]
  \[
  = \theta g(x_1, t_1) + (1 - \theta)g(x_2, t_2)
  \]

  the inequality follows from convexity of \(f\):
  \[
f(\mu(x_1/t_1) + (1 - \mu)(x_2/t_2)) \leq \mu f(x_1/t_1) + (1 - \mu) f(x_2/t_2)
  \]

  where \(\mu = \theta t_1 / (\theta t_1 + (1 - \theta)t_2)\)

Convex functions 3.26
The conjugate function

the **conjugate** of a function $f$ is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^T x - f(x))$$

- $f^*$ is convex (even if $f$ is not)
- will be useful when we discuss duality
Examples

- negative logarithm $f(x) = -\log x$
  
  $$f^*(y) = \sup_{x>0} (xy + \log x)$$
  
  $$= \begin{cases} 
  -1 - \log(-y) & y < 0 \\
  \infty & \text{otherwise}
  \end{cases}$$

- strictly convex quadratic $f(x) = (1/2)x^T Q x$ with $Q \in S^n_{++}$
  
  $$f^*(y) = \sup_x (y^T x - (1/2)x^T Q x)$$
  
  $$= \frac{1}{2} y^T Q^{-1} y$$
Quasiconvex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if $\text{dom } f$ is convex and the sublevel sets

$$S_\alpha = \{ x \in \text{dom } f \mid f(x) \leq \alpha \}$$

are convex for all $\alpha$

- $f$ is quasiconcave if $-f$ is quasiconvex
- $f$ is quasilinear if it is quasiconvex and quasiconcave

\[ S_\alpha = [a, b] \]
\[ S_\beta = (-\infty, c) \]
Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbb{R}$
- $\text{ceil}(x) = \inf\{z \in \mathbb{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbb{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbb{R}^2_{++}$
- linear-fractional function
  \[
f(x) = \frac{a^T x + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}\]
  is quasilinear
- distance ratio
  \[
f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \quad \text{dom } f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}\]
  is quasiconvex
Internal rate of return

• cash flow \( x = (x_0, \ldots, x_n) \); \( x_i \) is payment in period \( i \) (to us if \( x_i > 0 \))

• we assume \( x_0 < 0 \) and \( x_0 + x_1 + \cdots + x_n > 0 \)

• present value of cash flow \( x \), for interest rate \( r \):

\[
PV(x, r) = \sum_{i=0}^{n} (1 + r)^{-i} x_i
\]

• internal rate of return is smallest interest rate for which \( PV(x, r) = 0 \):

\[
IRR(x) = \inf\{ r \geq 0 \mid PV(x, r) = 0 \}
\]

IRR is quasiconcave: superlevel set is intersection of open halfspaces

\[
IRR(x) \geq R \quad \iff \quad \sum_{i=0}^{n} (1 + r)^{-i} x_i > 0 \quad \text{for } 0 \leq r < R
\]
Properties

Modified Jensen inequality: for quasiconvex $f$

$$0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\}$$

First-order condition: differentiable $f$ with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \implies \nabla f(x)^T (y - x) \leq 0$$

Sums: sums of quasiconvex functions are not necessarily quasiconvex
Log-concave and log-convex functions

A positive function $f$ is log-concave if $\log f$ is concave:

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

$f$ is log-convex if $\log f$ is convex

- powers: $x^a$ on $\mathbb{R}_{++}$ is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- many common probability densities are log-concave, e.g., normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1} (x-\bar{x})}$$

- cumulative Gaussian distribution function $\Phi$ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du$$
Properties of log-concave functions

- twice differentiable $f$ with convex domain is log-concave if and only if
  $$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T \quad \text{for all } x \in \text{dom } f$$

- product of log-concave functions is log-concave

- sum of log-concave functions is not always log-concave

- integration: if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is log-concave, then
  $$g(x) = \int f(x, y) \, dy$$
  is log-concave (not easy to show)
Consequences of integration property

- convolution \( f \ast g \) of log-concave functions \( f, g \) is log-concave

\[
(f \ast g)(x) = \int f(x - y)g(y)dy
\]

- if \( C \subseteq \mathbb{R}^n \) convex and \( y \) is a random variable with log-concave p.d.f. then

\[
f(x) = \text{prob}(x + y \in C)
\]

is log-concave

proof: write \( f(x) \) as integral of product of log-concave functions

\[
f(x) = \int g(x + y)p(y)dy, \quad g(u) = \begin{cases} 
1 & u \in C \\
0 & u \notin C
\end{cases}
\]

\( p \) is p.d.f. of \( y \)
Example: yield function

\[ Y(x) = \text{prob}(x + w \in S) \]

- \( x \in \mathbb{R}^n \): nominal parameter values for product
- \( w \in \mathbb{R}^n \): random variations of parameters in manufactured product
- \( S \): set of acceptable values

If \( S \) is convex and \( w \) has a log-concave p.d.f., then

- \( Y \) is log-concave
- yield regions \( \{ x \mid Y(x) \geq \alpha \} \) are convex