2. Convex sets

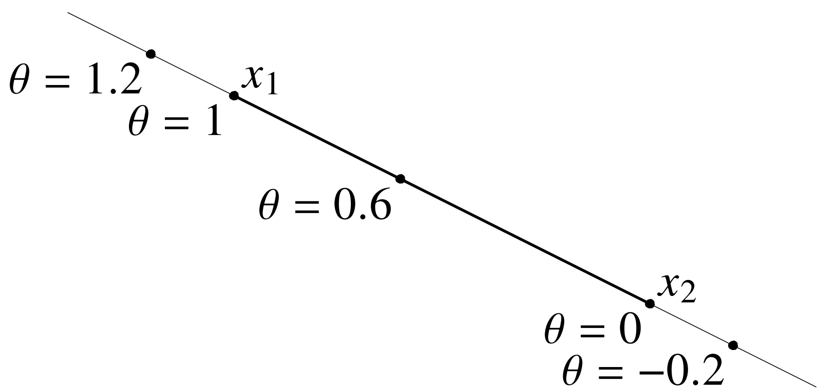
- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- dual cones
- separating and supporting hyperplanes

Affine set

Line through points x_1, x_2: all points

$$x = \theta x_1 + (1 - \theta) x_2 \quad \text{with } \theta \in \mathbf{R}$$

x is a called an *affine combination* of x_1 and x_2



Affine set: a set that contains the line through any two distinct points in the set

Example: the solution set of linear equations $\{x \mid Ax = b\}$ is an affine set

conversely, every affine set can be expressed as solution set of linear equations

Convex set

Line segment between points x_1, x_2: all points

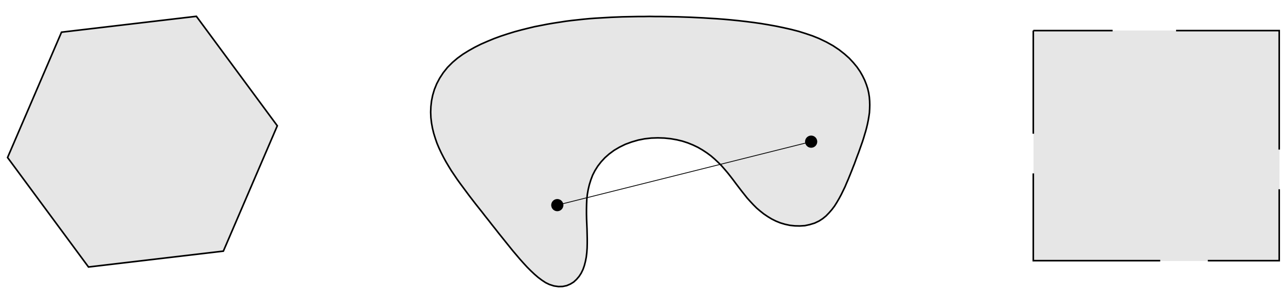
$$x = \theta x_1 + (1 - \theta) x_2 \quad \text{with } 0 \le \theta \le 1$$

x is a called a *convex combination* of x_1 and x_2

Convex set: a set that contains the line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

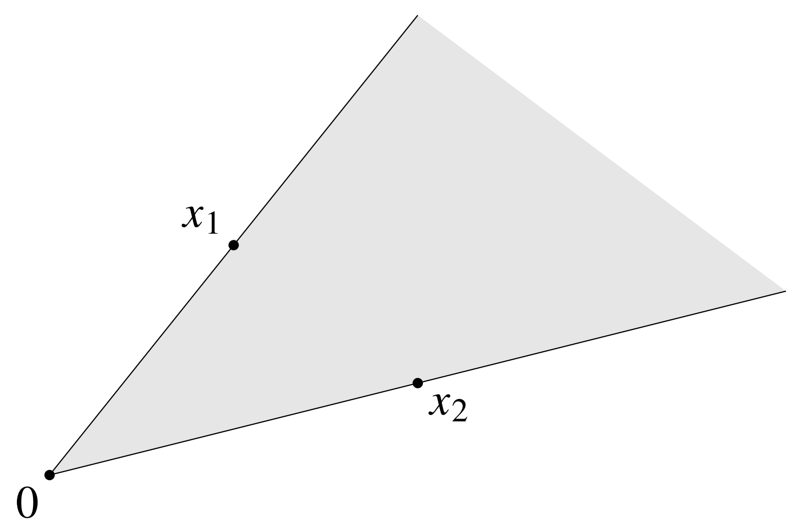
Examples (one convex, two nonconvex sets)



Convex cone

Conic (nonnegative) combination of points x_1, x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 \quad \text{with } \theta_1 \geq 0, \theta_2 \geq 0$$



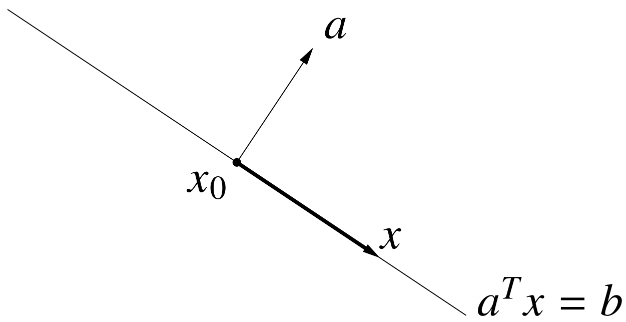
Convex cone: a set that contains all conic combinations of points in the set

Common convex sets

- hyperplanes and halfspaces
- Euclidean balls and ellipsoids
- norm balls and norm cones
- polyhedra
- positive semidefinite cone

Hyperplanes and halfspaces

Hyperplane: set of the form $\{x \mid a^T x = b\}$ where $a \neq 0$

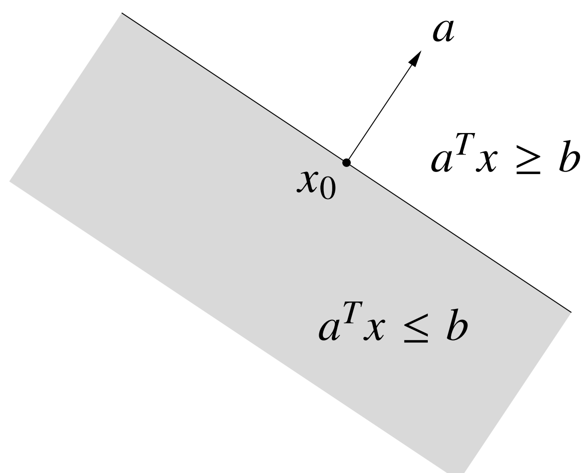


x_0 is a particular element, *e.g.*,

$$x_0 = \frac{b}{a^T a} a$$

$a^T x = b$ if and only if $a \perp (x - x_0)$

Halfspace: set of the form $\{x \mid a^T x \leq b\}$ where $a \neq 0$



hyperplanes are affine and convex; halfspaces are convex

Euclidean balls and ellipsoids

(Euclidean) ball with center x_c and radius r:

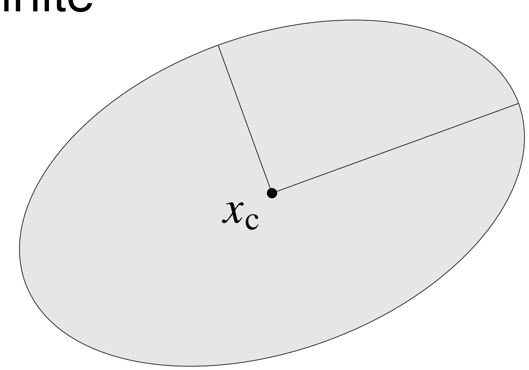
$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

$\|\cdot\|_2$ denotes the Euclidean norm

Ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$$

with P symmetric positive definite



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Principal axes

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

Eigendecomposition: $P = Q \Lambda Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T$

- Q is orthogonal ($Q^T = Q^{-1}$) with columns q_i
- Λ is diagonal with diagonal elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$

Change of variables: $y = Q^T (x - x_c)$, $x = x_c + Qy$

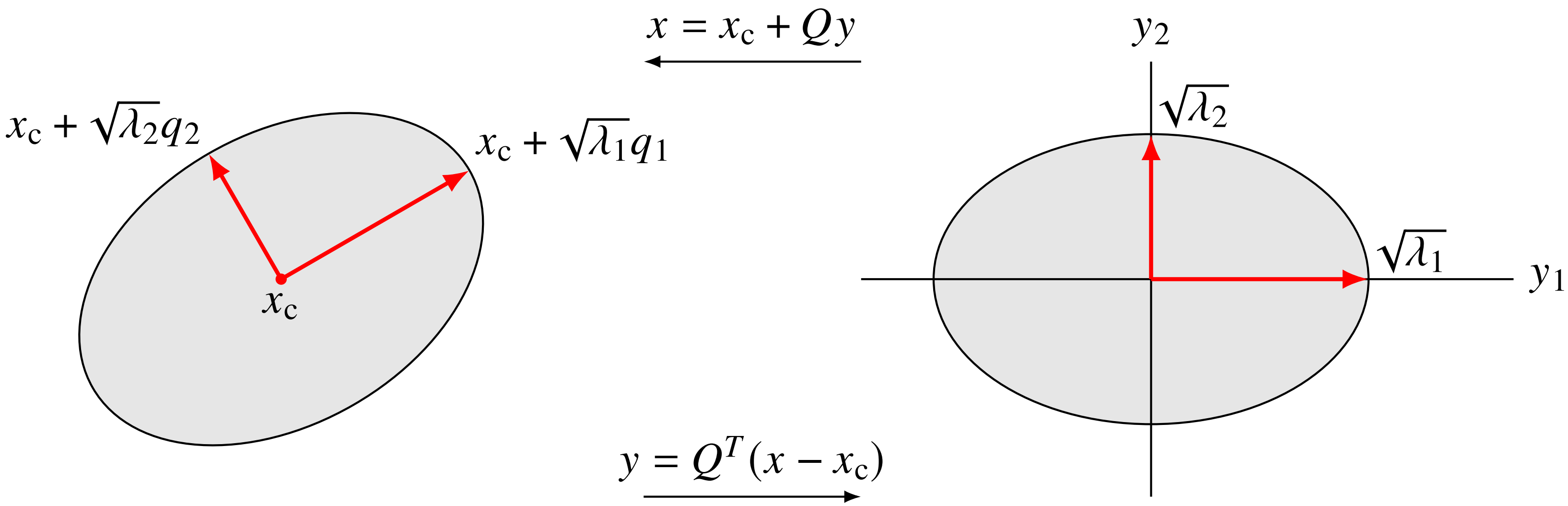
- after the change of variables the ellipsoid is described by

$$y^T \Lambda^{-1} y = \frac{y_1^2}{\lambda_1} + \cdots + \frac{y_n^2}{\lambda_n} \leq 1$$

  this is an ellipsoid centered at the origin, and aligned with the coordinate axes

- eigenvectors q_i of P give the principal axes of $\mathcal{E}$
- the width of $\mathcal{E}$ along the principal axis corresponding to q_i is $2\sqrt{\lambda_i}$

Example in $\mathbf{R}^2$



Exercise: give an interpretation of $\mathrm{tr}(P)$ as a measure of the size of the ellipsoid

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

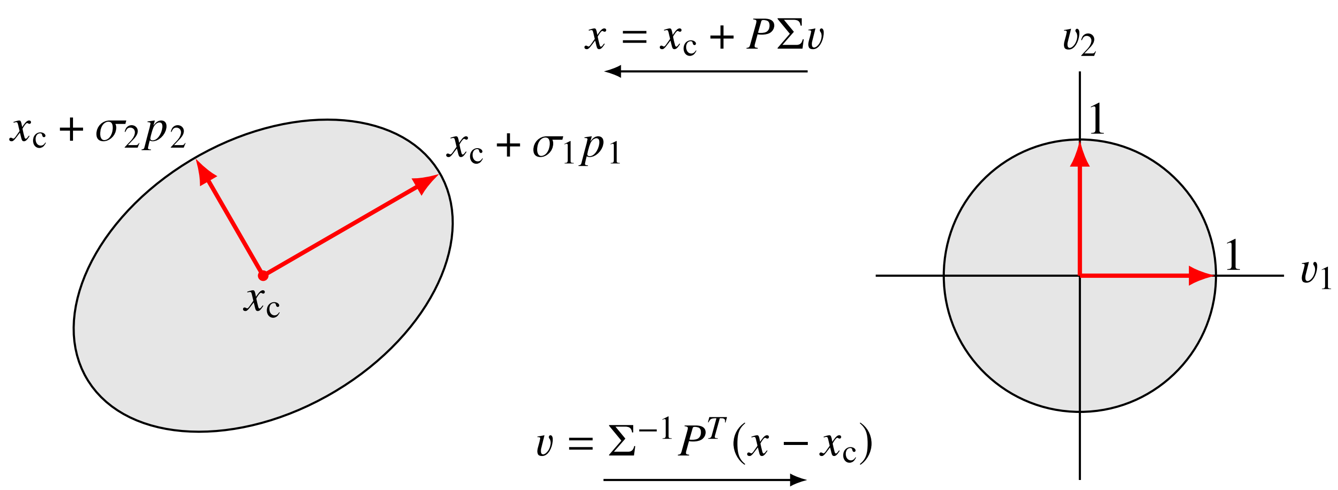
Principal axes (second representation)

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \leq 1\}$$

Singular value decomposition

$$A = P\Sigma Q^T = \sum_{i=1}^{n} \sigma_i p_i q_i^T$$

- P, Q orthogonal; Σ diagonal with diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$
- since $\|Q^T u\|_2 = \|u\|_2$ for orthogonal Q, we have $\mathcal{E} = \{x_c + P\Sigma v \mid \|v\|_2 \leq 1\}$



Norms

Norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$ for all x
- $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

Notation

- $\|\cdot\|$ is a general (unspecified) norm
- $\|\cdot\|_{\mathrm{symb}}$ is a particular norm

Common vector norms

for $x \in \mathbf{R}^n$

- Euclidean norm

$$\|x\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$$

- p-norm ($p \geq 1$)

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

- Chebyshev norm (∞-norm)

$$\|x\|_\infty = \max_{k=1,\ldots,n} |x_k|$$

- quadratic norm

$$\|x\|_A = (x^T A x)^{1/2}$$

  with A symmetric positive definite

Common matrix norms

for $X \in \mathbf{R}^{m \times n}$

- Frobenius norm

$$\|X\|_F = (\sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}^2)^{1/2}$$

- 2-norm (spectral norm, operator norm)

$$\|X\|_2 = \sup_{y \neq 0} \frac{\|Xy\|_2}{\|y\|_2} = \sigma_{\max}(X)$$

$\sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$ is largest singular value of X

Norm balls and norm cones

Norm ball with center x_c and radius r:

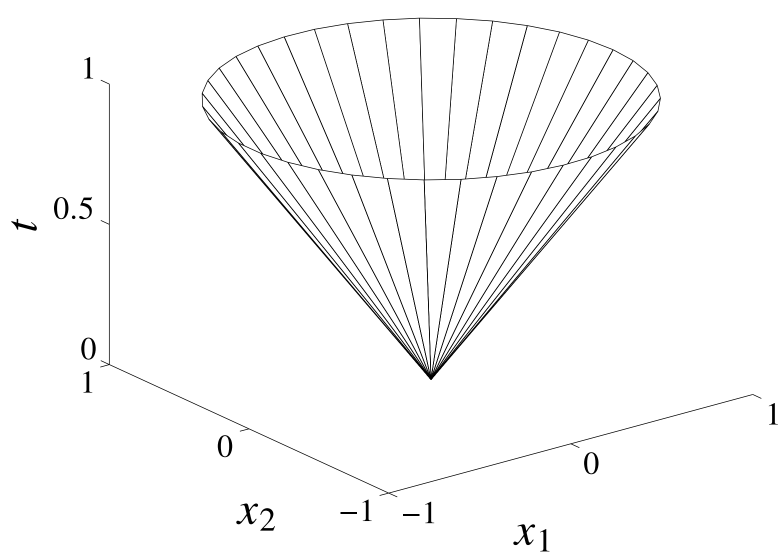
$$\{x \mid \|x - x_c\| \leq r\}$$

norm balls are convex sets

Norm cone:

$$\{(x, t) \mid \|x\| \leq t\}$$

- norm cones are convex cones
- example: second order cone (norm cone for Euclidean norm)

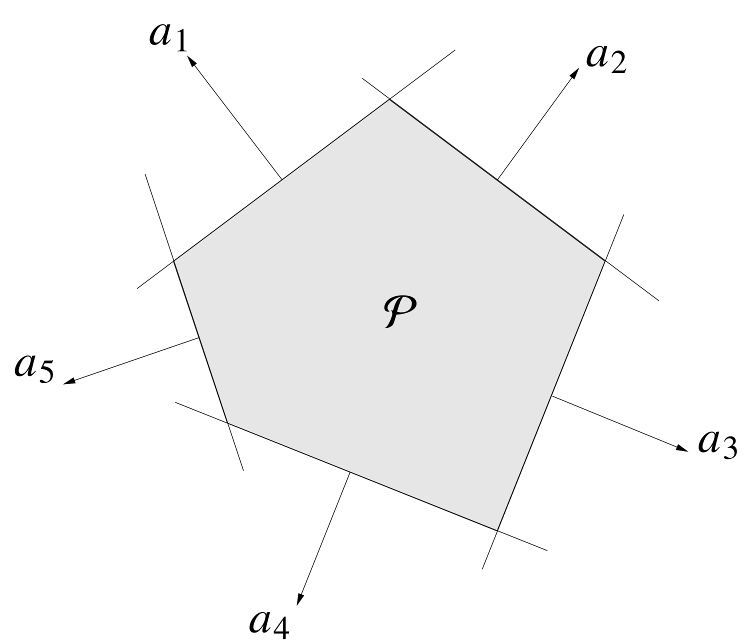


Polyhedra

Polyhedron: solution set of *finitely many* linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

$\preceq$ denotes componentwise inequality between vectors



a polyhedron is the intersection of a finite number of halfspaces and hyperplanes

Positive semidefinite cone

Notation

- $\mathbf{S}^n$ is the set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: the set of positive semidefinite $n \times n$ matrices

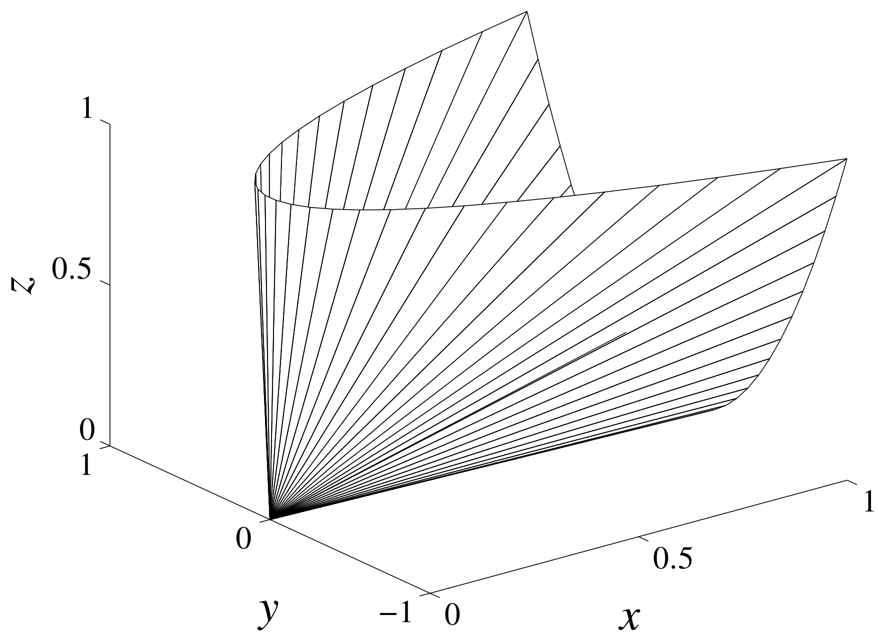
$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: the positive definite $n \times n$ matrices

Example

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$$



Operations that preserve convexity

methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

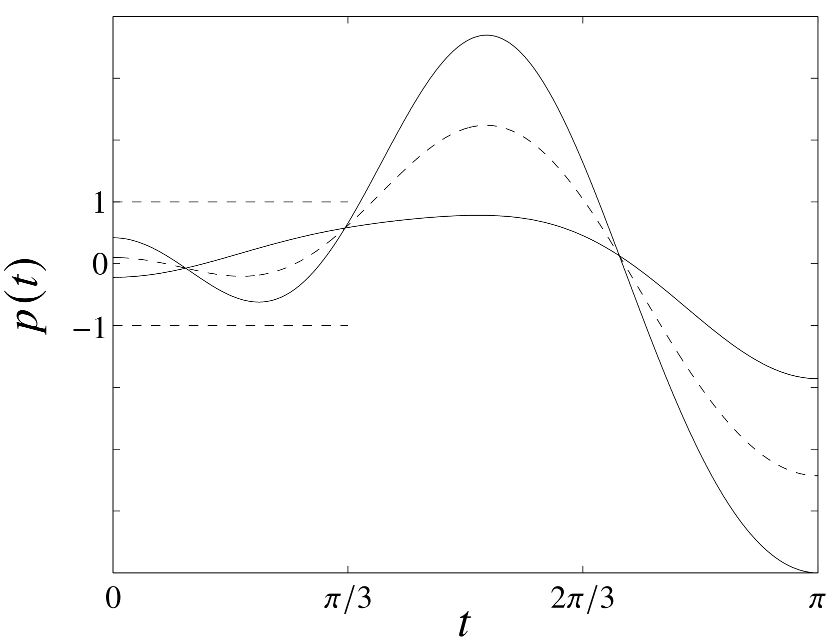
the intersection of (any number of) convex sets is convex

Example

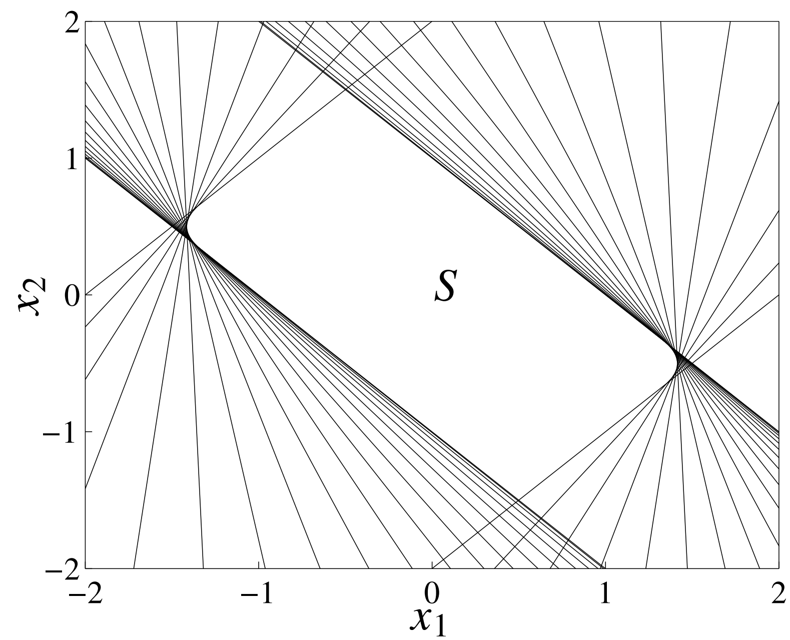
$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

three elements of S for $m = 4$



the set S for $m = 2$



Convex combination and convex hull

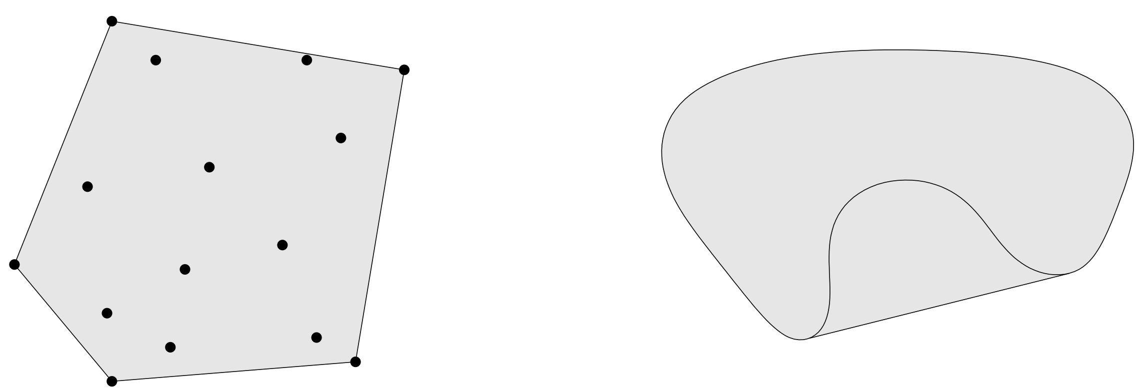
Convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

Convex hull (of a set S)

- $\mathbf{conv}\,(S)$ is set of all convex combinations of points in S
- $\mathbf{conv}\,(S)$ is the intersection of all convex sets that contain S



Affine function

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is an affine function:

$$f(x) = Ax + b$$

with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \quad \text{convex} \qquad \Longrightarrow \qquad f(S) = \{Ax + b \mid x \in S\} \quad \text{is convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \quad \text{convex} \qquad \Longrightarrow \qquad f^{-1}(C) = \{x \in \mathbf{R}^n \mid Ax + b \in C\} \quad \text{is convex}$$

Exercise

prove the statements on page 2.20

Solution (image of convex set under f is convex)

- suppose $S \subseteq \mathbf{R}^n$ is convex and consider two points $y_1, y_2 \in f(S)$:

$$y_1 = Ax_1 + b, \quad y_2 = Ax_2 + b \quad \text{where } x_1, x_2 \in S$$

- consider convex combination $y = \theta y_1 + (1-\theta)y_2$:

$$\begin{aligned} y &= \theta y_1 + (1-\theta)y_2 \\ &= \theta(Ax_1 + b) + (1-\theta)(Ax_2 + b) \\ &= A(\theta x_1 + (1-\theta)x_2) + b \\ &= Ax + b \end{aligned}$$

  where $x = \theta x_1 + (1-\theta)x_2)$

- $x \in S$ because S is convex, so $y = Ax + b \in f(S)$

Examples

- scaling, translation, projection
- image and inverse image of norm ball under affine transformation

$$\{Ax + b \mid \|x\| \leq 1\}, \qquad \{x \mid \|Ax + b\| \leq 1\}$$

- hyperbolic cone

$$\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\} \qquad \text{with } P \in \mathbf{S}^n_+$$

- solution set of linear matrix inequality

$$\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\} \qquad \text{with } A_i, B \in \mathbf{S}^p$$

Perspective and linear-fractional function

Perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

Linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

- the composition of the perspective function and an affine function
- image, inverse image of convex sets under linear-fractional function are convex

Exercise

prove that images/inverse images of convex sets under perspective are convex

Solution (image of convex set under perspective)

- suppose $S \subseteq \mathbf{R}^{n+1}$ is convex and consider two points $y_1, y_2 \in P(S)$:

$$y_1 = x_1/t_1, \quad y_2 = x_2/t_2 \quad \text{where } (x_1, t_1), (x_2, t_2) \in S \text{ and } t_1, t_2 > 0$$

- consider convex combination $y = \theta y_1 + (1 - \theta) y_2$ and verify that

$$y = \theta(x_1/t_1) + (1 - \theta)(x_2/t_2) = \frac{\mu x_1 + (1 - \mu) x_2}{\mu t_1 + (1 - \mu) t_2}$$

where

$$\mu = \frac{\theta/t_1}{\theta/t_1 + (1 - \theta)/t_2}, \qquad 1 - \mu = \frac{(1 - \theta)/t_2}{\theta/t_1 + (1 - \theta)/t_2}$$

- this shows that y is the perspective x/t of the convex combination

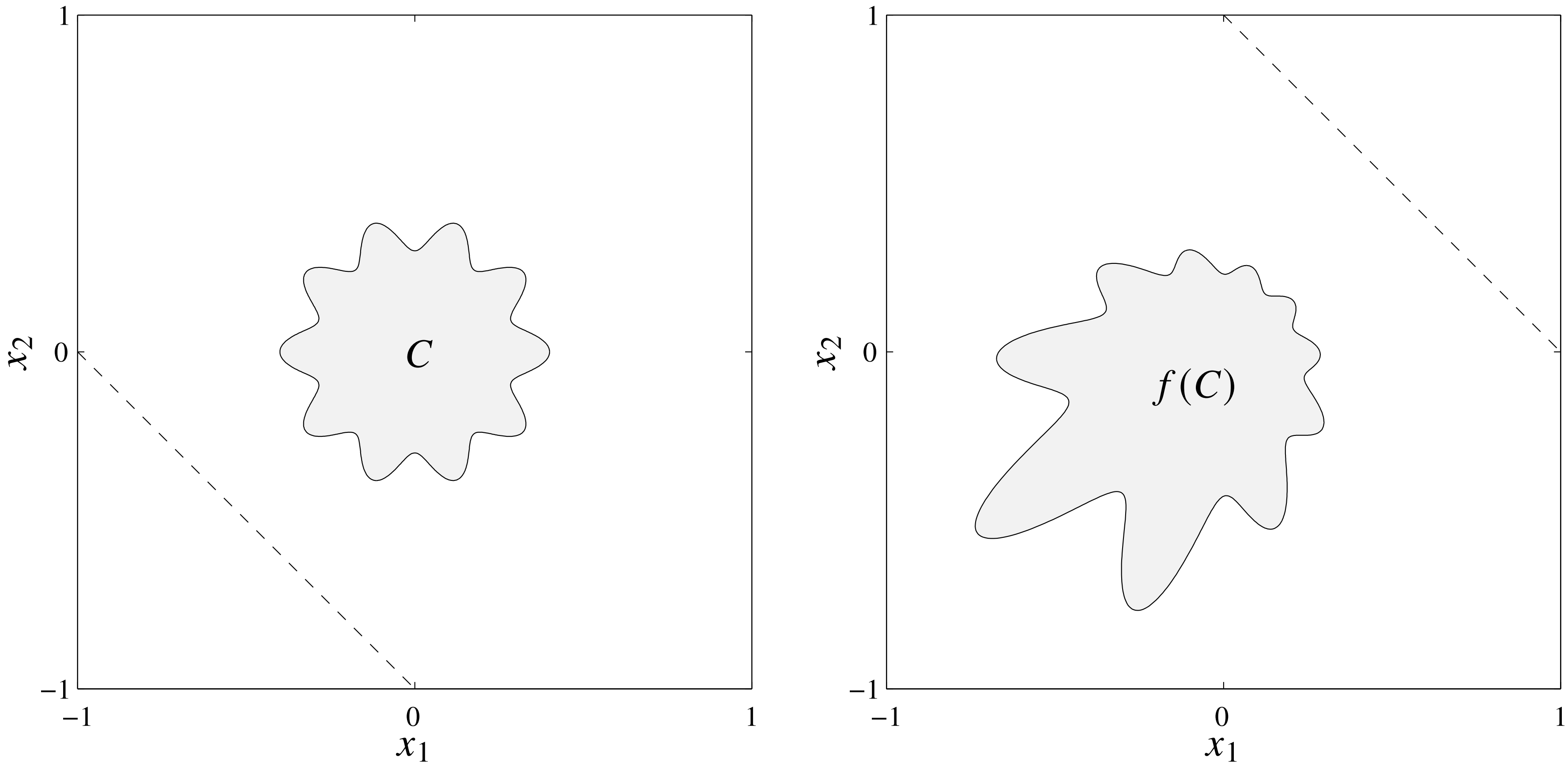
$$(x, t) = \mu(x_1, t_1) + (1 - \mu)(x_2, t_2)$$

$(x, t) \in S$ by convexity of S, so $y = x/t \in P(S)$

Example

a linear-fractional function from $\mathbf{R}^2$ to $\mathbf{R}^2$

$$f(x) = \frac{1}{x_1 + x_2 + 1} x, \qquad \mathbf{dom}\, f = \{(x_1, x_2) \mid x_1 + x_2 + 1 > 0\}$$



Proper cone

Proper cone: a convex cone $K \subseteq \mathbf{R}^n$ that satisfies three properties

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Examples

- nonnegative orthant

$$K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\}$$

- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

Generalized inequality

Generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \operatorname{int} K$$

Examples

- componentwise inequality ($K = \mathbf{R}_+^n$)

$$x \preceq_{\mathbf{R}_+^n} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}_+^n$)

$$X \preceq_{\mathbf{S}_+^n} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

Properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

Minimum and minimal elements

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

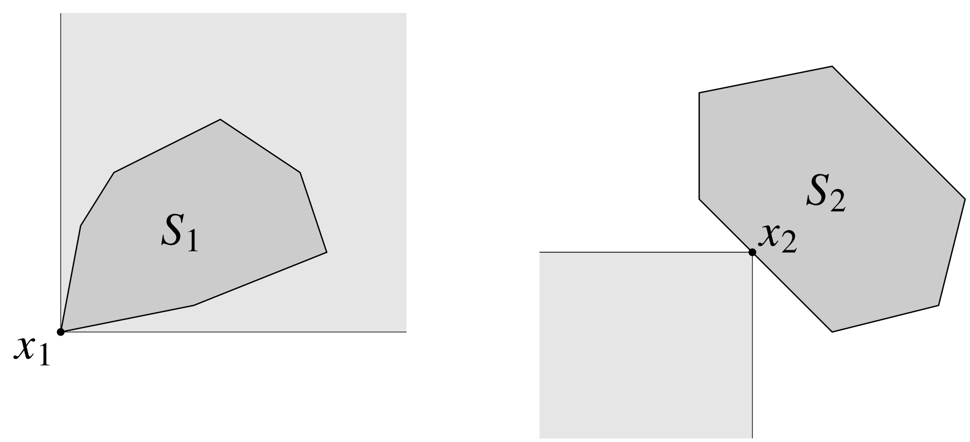
$$y \in S \quad \Longrightarrow \quad x \preceq_K y$$

$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \quad \Longrightarrow \quad y = x$$

Example ($K = \mathbf{R}^2_+$)

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2



Inner products

in this course we will use the following standard inner products

- for vectors $x, y \in \mathbf{R}^n$:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n = x^T y$$

- for matrices $X, Y \in \mathbf{R}^{m \times n}$

$$\langle X, Y \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij} = \mathrm{tr}(X^T Y)$$

- for symmetric matrices $X, Y \in \mathbf{S}^n$

$$\langle X, Y \rangle = \sum_{i=1}^{n} X_{ii} Y_{ii} + 2 \sum_{i>j} X_{ij} Y_{ij} = \mathrm{tr}(XY)$$

Dual cones

Dual cone of a cone K:

$$K^* = \{y \mid \langle y, x \rangle \geq 0 \text{ for all } x \in K\}$$

note: definition depends on choice of inner product

Examples

	K	K^*
nonnegative orthant	$\mathbf{R}^n_+$	$\mathbf{R}^n_+$
second order cone	$\{(x,t) \mid \|x\|_2 \leq t\}$	$\{(x,t) \mid \|x\|_2 \leq t\}$
1-norm cone	$\{(x,t) \mid \|x\|_1 \leq t\}$	$\{(x,t) \mid \|x\|_\infty \leq t\}$
positive semidefinite cone	$\mathbf{S}^n_+$	$\mathbf{S}^n_+$

three of the four examples are *self-dual* ($K^* = K$)

Exercise

derive the duals of the four examples on page 2.30

Solution ($K = \mathbf{R}^n_+$ is self-dual for inner product $\langle y, x \rangle = y^T x$)

- suppose $y \succeq 0$; then $y \in K^*$ because

$$y_1 x_1 + \cdots + y_n x_n \geq 0 \quad \text{for all } x \succeq 0$$

- suppose $y \not\succeq 0$; this means that $y_k < 0$ for some k

  let x be the kth standard unit vector:

$$x_i = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

  then $x \in K$ but the inner product $y^T x = y_k$ is negative; therefore $y \notin K^*$

Exercise

Solution ($K = \{(x,t) \mid \|x\|_2 \le t\}$ is self-dual)

- suppose $\|y\|_2 \le s$; then $(y,s) \in K^*$ because for all $(x,t) \in K$,

$$\begin{aligned}\left\langle \begin{bmatrix} y \\ s \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle &= y_1x_1 + \cdots + y_nx_n + st \\ &\ge -\|y\|_2\|x\|_2 + st \qquad \text{(by Cauchy–Schwarz inequality)} \\ &\ge s(t - \|x\|_2) \\ &\ge 0\end{aligned}$$

- suppose $\|y\|_2 > s$

  define $x = -y/\|y\|_2$ and $t = 1$; then $(x,t) \in K$ but

$$\left\langle \begin{bmatrix} y \\ s \end{bmatrix}, \begin{bmatrix} x \\ t \end{bmatrix} \right\rangle = y^Tx + st = -\|y\|_2 + s < 0$$

  therefore $(y,s) \notin K^*$

Exercise

Solution ($K = \mathbf{S}^n_+$ is self-dual for inner product $\langle Y, X \rangle = \mathrm{tr}(YX)$)

- suppose $Y \succeq 0$ with eigendecomposition

$$Y = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

then $Y \in K^*$ because for all $X \succeq 0$,

$$\begin{aligned} \mathrm{tr}(YX) &= \mathrm{tr}\left(\left(\sum_{i=1}^{n} \lambda_i q_i q_i^T\right) X\right) \\ &= \sum_{i=1}^{n} \lambda_i \,\mathrm{tr}(q_i q_i^T X) \\ &= \sum_{i=1}^{n} \lambda_i q_i^T X q_i \\ &\geq 0 \end{aligned}$$

(last step follows because $\lambda_i \geq 0$ and X is positive semidefinite)

- suppose $Y \not\succeq 0$, *i.e.*, there exists a vector a with $a^T Y a < 0$

  define $X = aa^T$; then $X \in \mathbf{S}^n_+$ but

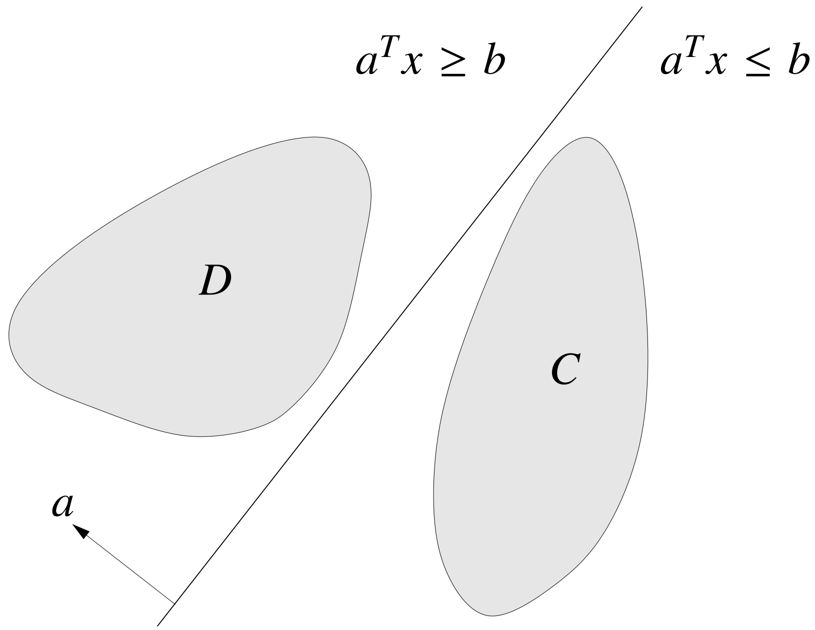
  $$\mathrm{tr}(YX) = \mathrm{tr}(Yaa^T) = a^T Y a < 0$$

  therefore $Y \notin K^*$

Separating hyperplane theorem

if C and D are nonempty disjoint convex sets, there exist $a \neq 0$, b s.t.

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

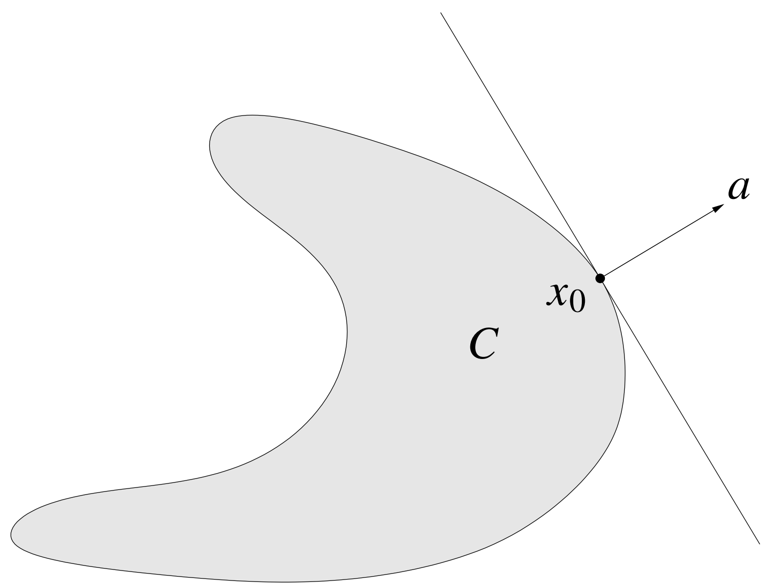
strict separation requires additional assumptions (*e.g.*, C closed, D a singleton)

Supporting hyperplane theorem

Supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



Supporting hyperplane theorem:

there exists a supporting hyperplane at every boundary point of a convex set C