# 7. Statistical estimation

- maximum likelihood estimation
- optimal detector design
- experiment design

# Parametric distribution estimation

- distribution estimation problem: estimate probability density $p(y)$ of a random variable from observed values
- parametric distribution estimation: choose from a family of densities $p_x(y)$, indexed by a parameter $x$

**maximum likelihood estimation**

$$\text{maximize (over } x\text{)} \quad \log p_x(y)$$

- $y$ is observed value
- $l(x) = \log p_x(y)$ is called log-likelihood function
- can add constraints $x \in C$ explicitly, or define $p_x(y) = 0$ for $x \notin C$
- a convex optimization problem if $\log p_x(y)$ is concave in $x$ for fixed $y$

# Linear measurements with IID noise

**linear measurement model**

$$y_i = a_i^T x + v_i, \quad i = 1, \ldots, m$$

- $x \in \mathbf{R}^n$ is vector of unknown parameters
- $v_i$ is IID measurement noise, with density $p(z)$
- $y_i$ is measurement: $y \in \mathbf{R}^m$ has density $p_x(y) = \prod_{i=1}^m p(y_i - a_i^T x)$

**maximum likelihood estimate:** any solution $x$ of

$$\text{maximize} \quad l(x) = \sum_{i=1}^m \log p(y_i - a_i^T x)$$

($y$ is observed value)

**examples**

- Gaussian noise $\mathcal{N}(0, \sigma^2)$: $p(z) = (2\pi\sigma^2)^{-1/2} e^{-z^2/(2\sigma^2)}$,

$$l(x) = -\frac{m}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{m}(a_i^T x - y_i)^2$$

  ML estimate is LS solution

- Laplacian noise: $p(z) = (1/(2a))e^{-|z|/a}$,

$$l(x) = -m\log(2a) - \frac{1}{a}\sum_{i=1}^{m}|a_i^T x - y_i|$$

  ML estimate is $\ell_1$-norm solution

- uniform noise on $[-a, a]$:

$$l(x) = \begin{cases} -m\log(2a) & |a_i^T x - y_i| \leq a, \quad i = 1, \ldots, m \\ -\infty & \text{otherwise} \end{cases}$$

  ML estimate is any $x$ with $|a_i^T x - y_i| \leq a$

# Logistic regression

random variable $y \in \{0, 1\}$ with distribution

$$p = \mathbf{prob}(y = 1) = \frac{\exp(a^T u + b)}{1 + \exp(a^T u + b)}$$

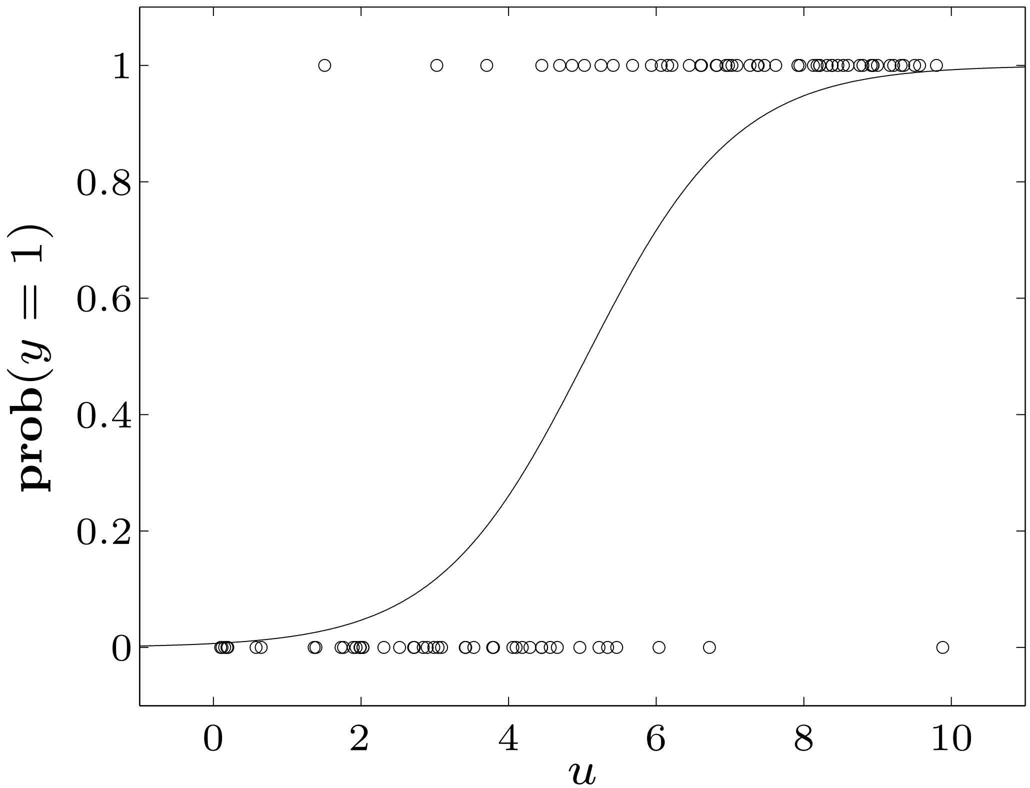
- $a$, $b$ are parameters; $u \in \mathbf{R}^n$ are (observable) explanatory variables
- estimation problem: estimate $a$, $b$ from $m$ observations $(u_i, y_i)$

**log-likelihood function** (for $y_1 = \cdots = y_k = 1$, $y_{k+1} = \cdots = y_m = 0$):

$$\begin{aligned} l(a, b) &= \log \left( \prod_{i=1}^{k} \frac{\exp(a^T u_i + b)}{1 + \exp(a^T u_i + b)} \prod_{i=k+1}^{m} \frac{1}{1 + \exp(a^T u_i + b)} \right) \\ &= \sum_{i=1}^{k} (a^T u_i + b) - \sum_{i=1}^{m} \log(1 + \exp(a^T u_i + b)) \end{aligned}$$

concave in $a$, $b$

**example** ($n = 1$, $m = 50$ measurements)




- circles show 50 points $(u_i, y_i)$
- solid curve is ML estimate of $p = \exp(au + b)/(1 + \exp(au + b))$

# (Binary) hypothesis testing

## detection (hypothesis testing) problem

given observation of a random variable $X \in \{1, \ldots, n\}$, choose between:

- hypothesis 1: $X$ was generated by distribution $p = (p_1, \ldots, p_n)$
- hypothesis 2: $X$ was generated by distribution $q = (q_1, \ldots, q_n)$

## randomized detector

- a nonnegative matrix $T \in \mathbf{R}^{2 \times n}$, with $\mathbf{1}^T T = \mathbf{1}^T$
- if we observe $X = k$, we choose hypothesis 1 with probability $t_{1k}$, hypothesis 2 with probability $t_{2k}$
- if all elements of $T$ are $0$ or $1$, it is called a deterministic detector

**detection probability matrix:**

$$D = \begin{bmatrix} Tp & Tq \end{bmatrix} = \begin{bmatrix} 1 - P_{\rm fp} & P_{\rm fn} \\ P_{\rm fp} & 1 - P_{\rm fn} \end{bmatrix}$$

- $P_{\rm fp}$ is probability of selecting hypothesis 2 if $X$ is generated by distribution 1 (false positive)
- $P_{\rm fn}$ is probability of selecting hypothesis 1 if $X$ is generated by distribution 2 (false negative)

**multicriterion formulation of detector design**

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{R}^2_+) & (P_{\rm fp}, P_{\rm fn}) = ((Tp)_2, (Tq)_1) \\ \text{subject to} & t_{1k} + t_{2k} = 1, \quad k = 1, \ldots, n \\ & t_{ik} \geq 0, \quad i = 1, 2, \quad k = 1, \ldots, n \end{array}$$

variable $T \in \mathbf{R}^{2\times n}$

**scalarization** (with weight $\lambda > 0$)

$$\begin{array}{ll} \text{minimize} & (Tp)_2 + \lambda(Tq)_1 \\ \text{subject to} & t_{1k} + t_{2k} = 1, \quad t_{ik} \geq 0, \quad i = 1, 2, \quad k = 1, \ldots, n \end{array}$$

an LP with a simple analytical solution

$$(t_{1k}, t_{2k}) = \begin{cases} (1, 0) & p_k \geq \lambda q_k \\ (0, 1) & p_k < \lambda q_k \end{cases}$$

- a deterministic detector, given by a likelihood ratio test
- if $p_k = \lambda q_k$ for some $k$, any value $0 \leq t_{1k} \leq 1$, $t_{1k} = 1 - t_{2k}$ is optimal ($i.e.$, Pareto-optimal detectors include non-deterministic detectors)

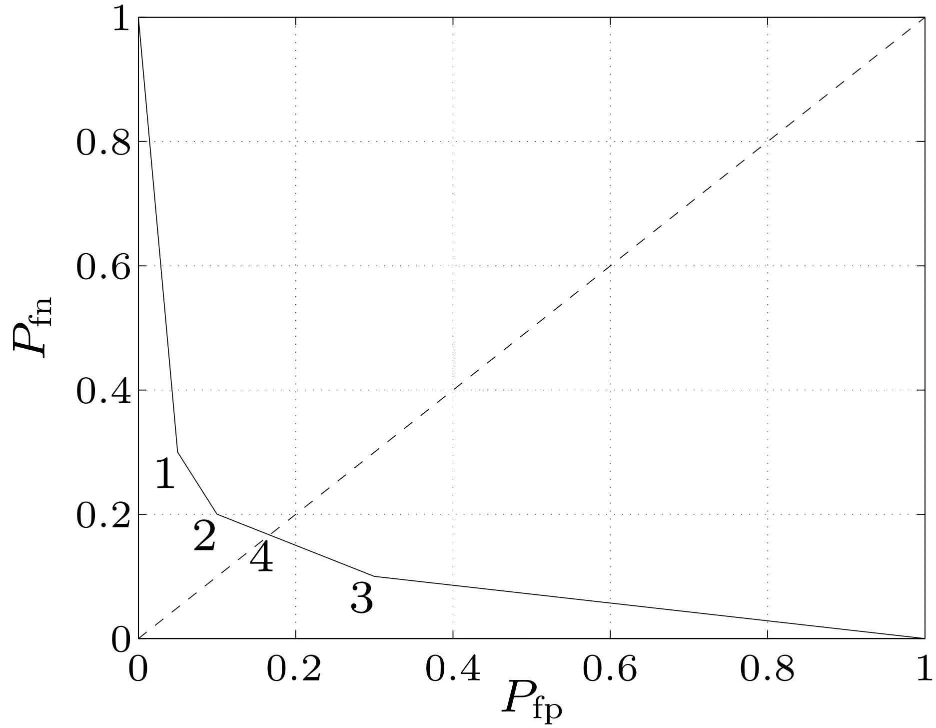
**minimax detector**

$$\begin{array}{ll} \text{minimize} & \max\{P_{\text{fp}}, P_{\text{fn}}\} = \max\{(Tp)_2, (Tq)_1\} \\ \text{subject to} & t_{1k} + t_{2k} = 1, \quad t_{ik} \geq 0, \quad i = 1, 2, \quad k = 1, \ldots, n \end{array}$$

an LP; solution is usually not deterministic

**example**

$$P = \begin{bmatrix} 0.70 & 0.10 \\ 0.20 & 0.10 \\ 0.05 & 0.70 \\ 0.05 & 0.10 \end{bmatrix}$$




solutions 1, 2, 3 (and endpoints) are deterministic; 4 is minimax detector

# Experiment design

$m$ linear measurements $y_i = a_i^T x + w_i$, $i = 1, \ldots, m$ of unknown $x \in \mathbf{R}^n$

- measurement errors $w_i$ are IID $\mathcal{N}(0, 1)$
- ML (least-squares) estimate is

$$\hat{x} = \left( \sum_{i=1}^{m} a_i a_i^T \right)^{-1} \sum_{i=1}^{m} y_i a_i$$

- error $e = \hat{x} - x$ has zero mean and covariance

$$E = \mathbf{E}\, e e^T = \left( \sum_{i=1}^{m} a_i a_i^T \right)^{-1}$$

confidence ellipsoids are given by $\{x \mid (x - \hat{x})^T E^{-1} (x - \hat{x}) \leq \beta\}$

**experiment design**: choose $a_i \in \{v_1, \ldots, v_p\}$ (a set of possible test vectors) to make $E$ 'small'

**vector optimization formulation**

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{S}_+^n) & E = \left(\sum_{k=1}^p m_k v_k v_k^T\right)^{-1} \\ \text{subject to} & m_k \geq 0, \quad m_1 + \cdots + m_p = m \\ & m_k \in \mathbf{Z} \end{array}$$

- variables are $m_k$ ($\#$ vectors $a_i$ equal to $v_k$)
- difficult in general, due to integer constraint

**relaxed experiment design**

assume $m \gg p$, use $\lambda_k = m_k/m$ as (continuous) real variable

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{S}_+^n) & E = (1/m)\left(\sum_{k=1}^p \lambda_k v_k v_k^T\right)^{-1} \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1 \end{array}$$

- common scalarizations: minimize $\log\det E$, $\mathbf{tr}\, E$, $\lambda_{\max}(E)$, . . .
- can add other convex constraints, *e.g.*, bound experiment cost $c^T\lambda \leq B$

## $D$-optimal design

$$\begin{array}{ll} \text{minimize} & \log\det\left(\sum_{k=1}^{p} \lambda_k v_k v_k^T\right)^{-1} \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1 \end{array}$$

interpretation: minimizes volume of confidence ellipsoids

**dual problem**

$$\begin{array}{ll} \text{maximize} & \log\det W + n\log n \\ \text{subject to} & v_k^T W v_k \leq 1, \quad k = 1,\ldots,p \end{array}$$

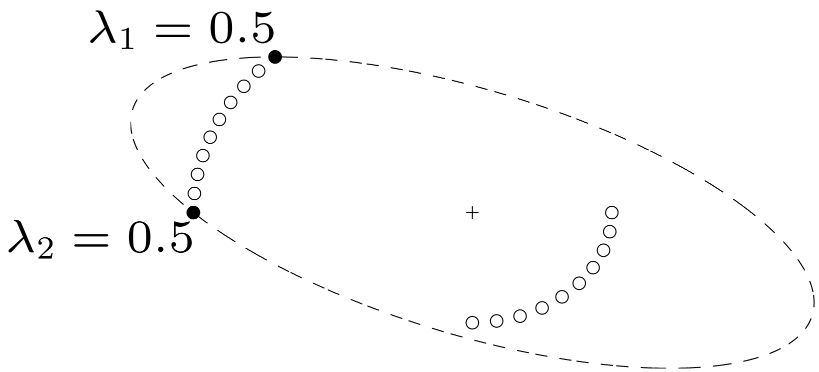
interpretation: $\{x \mid x^T W x \leq 1\}$ is minimum volume ellipsoid centered at origin, that includes all test vectors $v_k$

**complementary slackness:** for $\lambda$, $W$ primal and dual optimal

$$\lambda_k(1 - v_k^T W v_k) = 0, \quad k = 1,\ldots,p$$

optimal experiment uses vectors $v_k$ on boundary of ellipsoid defined by $W$

**example** $(p = 20)$




design uses two vectors, on boundary of ellipse defined by optimal $W$

**derivation of dual** of page 7–13

first reformulate primal problem with new variable $X$:

$$\begin{array}{ll} \text{minimize} & \log\det X^{-1} \\ \text{subject to} & X = \sum_{k=1}^p \lambda_k v_k v_k^T, \quad \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1 \end{array}$$

$$L(X, \lambda, Z, z, \nu) = \log\det X^{-1} + \mathbf{tr}\left(Z\left(X - \sum_{k=1}^p \lambda_k v_k v_k^T\right)\right) - z^T\lambda + \nu(\mathbf{1}^T\lambda - 1)$$

- minimize over $X$ by setting gradient to zero: $-X^{-1} + Z = 0$
- minimum over $\lambda_k$ is $-\infty$ unless $-v_k^T Z v_k - z_k + \nu = 0$

dual problem

$$\begin{array}{ll} \text{maximize} & n + \log\det Z - \nu \\ \text{subject to} & v_k^T Z v_k \le \nu, \quad k = 1, \ldots, p \end{array}$$

change variable $W = Z/\nu$, and optimize over $\nu$ to get dual of page 7–13