7. Statistical estimation

- maximum likelihood estimation
- optimal detector design
- experiment design
Parametric distribution estimation

- distribution estimation problem: estimate probability density $p(y)$ of a random variable from observed values
- parametric distribution estimation: choose from a family of densities $p_x(y)$, indexed by a parameter $x$

Maximum likelihood estimation

$$\text{maximize (over } x \text{) } \log p_x(y)$$

- $y$ is observed value
- $l(x) = \log p_x(y)$ is called log-likelihood function
- can add constraints $x \in C$ explicitly, or define $p_x(y) = 0$ for $x \notin C$
- a convex optimization problem if $\log p_x(y)$ is concave in $x$ for fixed $y$
Linear measurements with IID noise

Linear measurement model

\[ y_i = a_i^T x + v_i, \quad i = 1, \ldots, m \]

- \( x \in \mathbb{R}^n \) is vector of unknown parameters
- \( v_i \) is IID measurement noise, with density \( p(z) \)
- \( y_i \) is measurement: \( y \in \mathbb{R}^m \) has density

\[ p_x(y) = \prod_{i=1}^{m} p(y_i - a_i^T x) \]

**Maximum likelihood estimate:** any solution \( x \) of

\[
\text{maximize} \quad l(x) = \sum_{i=1}^{m} \log p(y_i - a_i^T x)
\]

(\( y \) is observed value)
Examples

- Gaussian noise $N(0, \sigma^2)$: $p(z) = (2\pi\sigma^2)^{-1/2} e^{-z^2/(2\sigma^2)}$,

$$l(x) = -\frac{m}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{m} (a_i^T x - y_i)^2$$

ML estimate is LS solution

- Laplacian noise: $p(z) = (1/(2a)) e^{-|z|/a}$,

$$l(x) = -m \log(2a) - \frac{1}{a} \sum_{i=1}^{m} |a_i^T x - y_i|$$

ML estimate is $\ell_1$-norm solution

- uniform noise on $[-a, a]$:

$$l(x) = \begin{cases} 
-m \log(2a) & |a_i^T x - y_i| \leq a, \quad i = 1, \ldots, m \\
-\infty & \text{otherwise}
\end{cases}$$

ML estimate is any $x$ with $|a_i^T x - y_i| \leq a$
Logistic regression

random variable $y \in \{0, 1\}$ with distribution

$$p = \text{prob}(y = 1) = \frac{\exp(a^T u + b)}{1 + \exp(a^T u + b)}$$

- $a, b$ are parameters; $u \in \mathbb{R}^n$ are (observable) explanatory variables
- estimation problem: estimate $a, b$ from $m$ observations $(u_i, y_i)$

Log-likelihood function (for $y_1 = \cdots = y_k = 1, y_{k+1} = \cdots = y_m = 0$):

$$l(a, b) = \log \left( \prod_{i=1}^{k} \frac{\exp(a^T u_i + b)}{1 + \exp(a^T u_i + b)} \prod_{i=k+1}^{m} \frac{1}{1 + \exp(a^T u_i + b)} \right)$$

$$= \sum_{i=1}^{k} (a^T u_i + b) - \sum_{i=1}^{m} \log(1 + \exp(a^T u_i + b))$$

concave in $a, b$
Example \((n = 1, m = 50 \text{ measurements})\)

- circles show 50 points \((u_i, y_i)\)
- solid curve is ML estimate of \(p = \exp(au + b)/(1 + \exp(au + b))\)
(Binary) hypothesis testing

Detection (hypothesis testing) problem

given observation of a random variable $X \in \{1, \ldots, n\}$, choose between:

- hypothesis 1: $X$ was generated by distribution $p = (p_1, \ldots, p_n)$
- hypothesis 2: $X$ was generated by distribution $q = (q_1, \ldots, q_n)$

Randomized detector

- a nonnegative matrix $T \in \mathbb{R}^{2 \times n}$, with $1^T T = 1^T$
- if we observe $X = k$, we choose hypothesis 1 with probability $t_{1k}$, hypothesis 2 with probability $t_{2k}$
- if all elements of $T$ are 0 or 1, it is called a deterministic detector
Detection probability matrix

\[ D = \begin{bmatrix} Tp & Tq \end{bmatrix} = \begin{bmatrix} 1 - P_{fp} & P_{fn} \\ P_{fp} & 1 - P_{fn} \end{bmatrix} \]

- \( P_{fp} \) is probability of selecting hypothesis 2 if \( X \) is generated by distribution 1 (false positive)
- \( P_{fn} \) is probability of selecting hypothesis 1 if \( X \) is generated by distribution 2 (false negative)

Multicriterion formulation of detector design

\[
\begin{align*}
\text{minimize (w.r.t. } R^2_+) & \quad (P_{fp}, P_{fn}) = ((T p)_2, (T q)_1) \\
\text{subject to} & \quad t_{1k} + t_{2k} = 1, \quad k = 1, \ldots, n \\
& \quad t_{ik} \geq 0, \quad i = 1, 2, \quad k = 1, \ldots, n \\
\text{variable } T & \in R^{2 \times n}
\end{align*}
\]
Scalarization (with weight \( \lambda > 0 \))

\[
\text{minimize} \quad (T_1 p)_2 + \lambda (T_1 q)_1 \\
\text{subject to} \quad t_{1k} + t_{2k} = 1, \quad t_{ik} \geq 0, \quad i = 1, 2, \quad k = 1, \ldots, n
\]
an LP with a simple analytical solution

\[
(t_{1k}, t_{2k}) = \begin{cases} 
(1, 0) & p_k \geq \lambda q_k \\
(0, 1) & p_k < \lambda q_k
\end{cases}
\]

- a deterministic detector, given by a likelihood ratio test
- if \( p_k = \lambda q_k \) for some \( k \), any value \( 0 \leq t_{1k} \leq 1, t_{1k} = 1 - t_{2k} \) is optimal (i.e., Pareto-optimal detectors include non-deterministic detectors)

Minimax detector

\[
\text{minimize} \quad \max\{P_{fp}, P_{fn}\} = \max\{(T_1 p)_2, (T_1 q)_1\} \\
\text{subject to} \quad t_{1k} + t_{2k} = 1, \quad t_{ik} \geq 0, \quad i = 1, 2, \quad k = 1, \ldots, n
\]
an LP; solution is usually not deterministic
Example

\[
P = \begin{bmatrix}
0.70 & 0.10 \\
0.20 & 0.10 \\
0.05 & 0.70 \\
0.05 & 0.10
\end{bmatrix}
\]

solutions 1, 2, 3 (and endpoints) are deterministic; 4 is minimax detector
Experiment design

$m$ linear measurements $y_i = a_i^T x + w_i$, $i = 1, \ldots, m$ of unknown $x \in \mathbb{R}^n$

- measurement errors $w_i$ are IID $\mathcal{N}(0, 1)$
- ML (least-squares) estimate is

$$\hat{x} = \left( \sum_{i=1}^{m} a_i a_i^T \right)^{-1} \sum_{i=1}^{m} y_i a_i$$

- error $e = \hat{x} - x$ has zero mean and covariance

$$E = \mathbb{E} e e^T = \left( \sum_{i=1}^{m} a_i a_i^T \right)^{-1}$$

confidence ellipsoids are given by $\{ x \mid (x - \hat{x})^T E^{-1} (x - \hat{x}) \leq \beta \}$

Experiment design: choose $a_i \in \{ v_1, \ldots, v_p \}$ (a set of possible test vectors) to make $E$ ‘small’
Vector optimization formulation

\[ \begin{align*}
\text{minimize (w.r.t. } S_+^n) & \quad E = \left( \sum_{k=1}^{p} m_k v_k v_k^T \right)^{-1} \\
\text{subject to} & \quad m_k \geq 0, \quad m_1 + \cdots + m_p = m \\
& \quad m_k \in \mathbb{Z}
\end{align*} \]

- variables are \( m_k \) (# vectors \( a_i \) equal to \( v_k \))
- difficult in general, due to integer constraint

Relaxed experiment design

assume \( m \gg p \), use \( \lambda_k = m_k / m \) as (continuous) real variable

\[ \begin{align*}
\text{minimize (w.r.t. } S_+^n) & \quad E = \left( 1/m \right) \left( \sum_{k=1}^{p} \lambda_k v_k v_k^T \right)^{-1} \\
\text{subject to} & \quad \lambda \geq 0, \quad 1^T \lambda = 1
\end{align*} \]

- common scalarizations: minimize log det \( E \), tr \( E \), \( \lambda_{\max}(E) \), ...
- can add other convex constraints, e.g., bound experiment cost \( c^T \lambda \leq B \)
**D-optimal design**

\[
\text{minimize} \quad \log \det \left( \sum_{k=1}^{p} \lambda_k v_k v_k^T \right)^{-1} \\
\text{subject to} \quad \lambda \geq 0, \quad 1^T \lambda = 1
\]

**Interpretation:** minimizes volume of confidence ellipsoids

**Dual problem**

\[
\text{maximize} \quad \log \det W + n \log n \\
\text{subject to} \quad v_k^T W v_k \leq 1, \quad k = 1, \ldots, p
\]

**Interpretation:** \( \{x \mid x^T W x \leq 1\} \) is minimum volume ellipsoid centered at origin, that includes all test vectors \( v_k \)

**Complementary slackness:** for \( \lambda, W \) primal and dual optimal

\[
\lambda_k (1 - v_k^T W v_k) = 0, \quad k = 1, \ldots, p
\]

optimal experiment uses vectors \( v_k \) on boundary of ellipsoid defined by \( W \)
Example \((p = 20)\)

\[
\lambda_1 = 0.5 \\
\lambda_2 = 0.5
\]

design uses two vectors, on boundary of ellipse defined by optimal \(W\)
Derivation of dual of page 7.13

first reformulate primal problem with new variable $X$:

\[
\begin{align*}
\text{minimize} & \quad \log \det X^{-1} \\
\text{subject to} & \quad X = \sum_{k=1}^{P} \lambda_k v_k v_k^T, \quad \lambda \geq 0, \quad 1^T \lambda = 1
\end{align*}
\]

\[
L(X, \lambda, Z, z, \nu) = \log \det X^{-1} + \text{tr}(Z(X - \sum_{k=1}^{P} \lambda_k v_k v_k^T)) - z^T \lambda + \nu(1^T \lambda - 1)
\]

- minimize over $X$ by setting gradient to zero: $-X^{-1} + Z = 0$
- minimum over $\lambda_k$ is $-\infty$ unless $-v_k^T Z v_k - z_k + \nu = 0$

dual problem

\[
\begin{align*}
\text{maximize} & \quad n + \log \det Z - \nu \\
\text{subject to} & \quad v_k^T Z v_k \leq \nu, \quad k = 1, \ldots, p
\end{align*}
\]

change variable $W = Z/\nu$, and optimize over $\nu$ to get dual of page 7.13