Semidefinite Programming Duality and Linear Time-Invariant Systems*

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Abstract

Several important problems in control theory can be reformulated as semidefinite programming problems, i.e., minimization of a linear objective subject to Linear Matrix Inequality (LMI) constraints. From convex optimization duality theory, conditions for infeasibility of the LMIs as well as dual optimization problems can be formulated. These can in turn be reinterpreted in control or system theoretic terms, often yielding new results or new proofs for existing results from control theory. We explore such connections for a few problems associated with linear time-invariant systems.

Index Terms—Semidefinite programming, linear matrix inequality (LMI), convex duality, linear time-invariant systems.

1 Introduction

Over the past few years, convex optimization, and semidefinite programming[1] (SDP) in particular, have come to be recognized as valuable numerical tools for control system analysis and design. A number of publications can be found in the control literature that survey applications of SDP to the solution of system and control problems (see for example [1, 2, 3, 4, 5]). In parallel, there has been considerable recent research on algorithms and software for the numerical solution of SDPs (for surveys, see [6, 7, 8, 9, 10, 11, 12]). This interest was primarily motivated by applications of SDP in combinatorial optimization but, more recently, also by the applications in control.

Thus far, the application of SDP in systems and control has been mainly motivated by the possibilities it offers for the numerical solution of analysis and synthesis problems for which no

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[1]We use SDP to mean both "semidefinite programming", as well as a "semidefinite program", i.e., a semidefinite programming problem.

analytical solutions are known [13, 14, 15]. In this paper, we explore another application of SDP: We discuss the application of duality theory to obtain new theoretical insight or to provide new proofs to existing results from system and control theory. Specifically, we discuss the following applications of SDP duality.

- Theorems of alternatives provide systematic and unified proofs of *necessary and sufficient conditions for solvability* of LMIs. As examples, we investigate the conditions for the existence of feasible solutions to Lyapunov and Riccati inequalities. As a by-product, we obtain a simple new proof of the Kalman-Yakubovich-Popov lemma.

  Several of the results that we use from convex duality require technical conditions (so-called *constraint qualifications*). We show that for problems involving Riccati inequalities these constraint qualifications are related to controllability and observability. In particular, we will obtain a new criterion for the controllability of an LTI system realization.

- The optimal solution of an SDP is characterized by *necessary and sufficient optimality conditions* that involve the dual variables. As an example, we show that the properties of the solution of the LQR problem can be derived directly from the SDP optimality conditions.

- The dual problem associated with an SDP can be used to derive *lower bounds* on the optimal value. As an example, we give new easily computed bounds on the H_∞-norm of an LTI system, and a duality-based proof of the Enns-Glover lower bound.

Several researchers have recently applied notions from convex optimization duality toward the re-interpretation of existing results and the derivation of new results in system theory. Rantzer [16] uses ideas from convexity theory to give a new proof of the Kalman-Yakubovich-Popov Lemma. In [17], SDP duality is used to study the relationship between "mixed-μ" and its upper bound. Henrion and Meinsma [18] apply SDP to provide a new proof of a generalized form of Lyapunov's matrix inequality on the location of the eigenvalues of a matrix in some region of the complex plane. Yao, Zhao, and Zhang [19] apply SDP optimality conditions to derive properties of the optimal solution of a stochastic linear-quadratic control problem; see also [20]. Our work is similar in spirit to these; however, the scope of our paper is wider, as we present new proofs to (and in many cases generalize) some of the results in these papers.

Notation

R (**R**$_+$) denotes the set of real (nonnegative real) numbers. **C** denotes the set of complex numbers. $\Re(\cdot)$ and $\Im(\cdot)$ denote respectively the real and imaginary parts of a complex scalar, vector or matrix. The matrix inequalities $A > B$ and $A \geq B$ mean A and B are square, Hermitian, and that $A - B$ is positive definite and positive semi-definite, respectively. The inequality $A \gneqq 0$ means that the matrix A is positive semidefinite and nonzero. $\mathcal{S}^n$ denotes the set of Hermitian $n \times n$ matrices with an associated inner product $\langle \cdot, \cdot \rangle_{\mathcal{S}^n}$. While the development in the sequel are applicable to any inner product on $\mathcal{S}^n$, we will assume that the standard inner product, given by $\langle A, B \rangle_{\mathcal{S}^n} = Tr\, A^* B = Tr\, AB$ is in effect.

$\mathbf{L}_2$ is the Hilbert space of square-integrable signals defined over **R**$_+$ (see for example [21]). $\mathbf{L}_{2e}$ denotes the extended space associated with $\mathbf{L}_2$.

2 Duality

Suppose that $\mathcal{S}$ is a space of block diagonal Hermitian matrices with some given dimensions, $\mathcal{S} = \mathcal{S}^{n_1} \times \cdots \times \mathcal{S}^{n_L}$, and with inner product

$$\langle diag\,(A_1, \ldots, A_L), diag\,(B_1, \ldots, B_L) \rangle_{\mathcal{S}} = \sum_{k=1}^{L} Tr\, A_k B_k.$$

Suppose that $\mathcal{V}$ is a finite-dimensional vector space with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$, $\mathcal{A} : \mathcal{V} \to \mathcal{S}$ is a linear mapping and $A_0 \in \mathcal{S}$. Then, the inequality

$$\mathcal{A}(x) + A_0 \geq 0 \tag{1}$$

is called a *Linear Matrix Inequality* or LMI. We let $\mathcal{A}^{\text{adj}}$ denote the adjoint mapping of $\mathcal{A}$. That is, $\mathcal{A}^{\text{adj}} : \mathcal{S} \to \mathcal{V}$ such that for all $x \in \mathcal{V}$ and $Z \in \mathcal{S}$, $\langle \mathcal{A}(x), Z \rangle_{\mathcal{S}} = \langle x, \mathcal{A}^{\text{adj}}(Z) \rangle_{\mathcal{V}}$.

Remark 1 We note that all the results derived in the sequel hold if $\mathcal{S}$ is a space of block diagonal real symmetric (instead of Hermitian) matrices. This would be applicable when dealing with real data. $\diamond$

2.1 Theorems of alternatives

We first examine criteria for solvability of different types of LMIs. We consider the following three feasibility problems.

- *Strict feasibility:* there exists an $x \in \mathcal{V}$ with $\mathcal{A}(x) + A_0 > 0$.
- *Nonzero feasibility:* there exists an $x \in \mathcal{V}$ with $\mathcal{A}(x) + A_0 \gneq 0$.
- *Feasibility:* there exists an $x \in \mathcal{V}$ with $\mathcal{A}(x) + A_0 \geq 0$.

By properly choosing $\mathcal{A}$ we will be able to address a wide variety of LMI feasibility problems. For example, when $\mathcal{V} = \mathbf{R}^m$, we can express $\mathcal{A}$ as

$$\mathcal{A}(x) = x_1 A_1 + x_2 A_2 + \cdots + x_m A_m, \tag{2}$$

where $A_i \in \mathcal{S}$ are given. With this parametrization, the three problems described above reduce to the following three basic LMIs:

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \quad > \quad 0, \tag{3}$$

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \quad \gneq \quad 0, \tag{4}$$

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m \quad \geq \quad 0. \tag{5}$$

There exists a rich literature on theorems of alternatives for generalized inequalities (i.e., inequalities with respect to nonpolyhedral convex cones), and linear matrix inequalities in particular. For our purposes the following three theorems will be sufficient. Owing to space limitations, we state these theorems without proof; proofs can be found in [22]. We refer to [23, 24, 25, 26] for more background on theorems of alternative for nonpolyhedral cones, and to [27, 28, 29] for results on linear matrix inequalities. We note that these theorems are special cases of the more general Hahn-Banach separation theorem; see for example [30].

Theorem 1 (ALT 1) *Exactly one of the following statements is true.*

1. *There exists an $x \in \mathcal{V}$ with $\mathcal{A}(x) + A_0 > 0$.*
2. *There exists a $Z \in \mathcal{S}$ with $Z \gneq 0$, $\mathcal{A}^{\mathrm{adj}}(Z) = 0$, and $\langle A_0, Z \rangle_{\mathcal{S}} \leq 0$.*

Theorem ALT 1 is the first example of a *theorem of alternatives*. The two statements in the theorem are called *strong alternatives*, because exactly one of them is true.

Example 1 The adjoint $\mathcal{A}_0^{\mathrm{adj}} : \mathcal{S} \to \mathbf{R}^m$ of the mapping defined by (2) is given by

$$\mathcal{A}_0^{\mathrm{adj}}(Z) = \left[\begin{array}{cccc} \mathit{Tr}\, A_1 Z & \mathit{Tr}\, A_2 Z & \cdots & \mathit{Tr}\, A_m Z \end{array} \right]^T.$$

Theorem **ALT 1** therefore implies that either there exists $x \in \mathbf{R}^m$ such that LMI (3) holds, or there exists $Z \in \mathcal{S}$ with $Z \gneq 0$ such that $\mathit{Tr}\, A_i Z = 0$, $i = 1, 2, \ldots, m$, and $\mathit{Tr}\, A_0 Z \leq 0$. $\diamond$

Example 2 As an example of an application of Theorem **ALT 1** in matrix algebra, consider Finsler's Theorem [31, 32, 33], which states that given $F \in \mathbf{C}^{n\times m}$ with rank $r < n$, and $G \in \mathcal{S}^n$, the condition that there exists $\mu \in \mathbf{R}$ such that $\mu FF^* - G > 0$, is equivalent to $(F^\perp)^* G F^\perp < 0$, where $F^\perp$ is a full-rank matrix whose columns span the left nullspace of F, i.e., $(F^\perp)^* F = 0$.

Let $\mathcal{A} : \mathbf{R} \to \mathcal{S}^n$ be defined by $\mathcal{A}(\mu) = \mu FF^*$, and let $A_0 = -G$. Then $\mathcal{A}^{\mathrm{adj}} : \mathcal{S}^n \to \mathbf{R}$ is given by $\mathcal{A}^{\mathrm{adj}}(Z) = Tr\,(F^* Z F)$. Then, there does not exist $\mu \in \mathbf{R}$ such that $\mathcal{A}(\mu) + A_0 > 0$, if and only if there exists a $Z \in \mathcal{S}^n$ with $Z \gneqq 0$, $Tr\,(F^* Z F) = 0$, $Tr\,(ZG) \geq 0$. Factoring Z as $Z = \sum_{i=1}^k \lambda_i u_i u_i^*$, where $\lambda_i > 0$, we must have for some i, $u_i^* F = 0$ and $u_i^* G u_i \geq 0$, which immediately means $(F^\perp)^* G F^\perp < 0$ is violated. Conversely, if $(F^\perp)^* G F^\perp \not< 0$, then for some nonzero $u \in \mathbf{C}^n$, we must have $u^* (F^\perp)^* G F^\perp u \geq 0$. Then, with $Z = (F^\perp u)(F^\perp u)^*$, it is readily verified that $Z \gneqq 0$ and $Tr\,(F^* Z F) = 0$. ◇

Example 3 Let $A, B \in \mathcal{S}^n$. The following result is a version of the *S-procedure* [34, 35, 36, 1]. There exist $\tau_1, \tau_2 \in \mathbf{R}$ such that

$$\tau_1 > 0, \quad \tau_2 > 0, \quad \tau_1 A + \tau_2 B > 0, \tag{6}$$

if and only if there exists no $z \in \mathbf{C}^n$ satisfying

$$z \neq 0, \quad z^* A z \leq 0, \quad z^* B z \leq 0. \tag{7}$$

We can derive the result from Theorem **ALT1** as follows. Define $\mathcal{A} : \mathbf{R}^2 \to \mathcal{S}^n \times \mathbf{R} \times \mathbf{R}$ as

$$\mathcal{A}(\tau_1, \tau_2) = diag\,(\tau_1 A + \tau_2 B, \tau_1, \tau_2)$$

and take $A_0 = 0$. Theorem **ALT1** states that (6) is infeasible if and only if there exists $Z \in \mathcal{S}^n$ such that

$$Z \gneqq 0, \quad Tr\,AZ \leq 0, \quad Tr\,BZ \leq 0.$$

This turns out to be equivalent to the existence of a $z \in \mathbf{C}^n$ such that (7) holds. The equivalence readily follows from the fact that the field of values of a pair of Hermitian matrices, which is defined as

$$F(A,B) = \{(z^* A z, z^* B z) \mid z \in \mathbf{C}^n, z^* z = 1\} \subseteq \mathbf{R}^2,$$

is a convex set (see [37, p.86]), and therefore equal to its convex hull

$$\begin{aligned} F(A,B) &= \mathbf{Co}\, F(A,B) \\ &= \{(Tr\,AZ, Tr\,BZ) \mid Z \in \mathcal{S}^n, Z \geq 0, Tr\,Z = 1\}. \end{aligned}$$

$\diamond$

Theorem 2 (ALT 2) *At most one of the following statements is true.*

1. *There exists an* $x \in \mathcal{V}$ *with* $\mathcal{A}(x) + A_0 \gneqq 0$.
2. *There exists a* $Z \in \mathcal{S}$ *with* $Z > 0$, $\mathcal{A}^{\mathrm{adj}}(Z) = 0$, *and* $\langle A_0, Z \rangle_{\mathcal{S}} \leq 0$.

Moreover, if $A_0 = \mathcal{A}(x_0)$ *for some* $x_0 \in \mathcal{V}$, *or if there exists no* $x \in \mathcal{V}$ *with* $\mathcal{A}(x) \gneqq 0$, *then exactly one of the two statements is true.*

The theorem gives a pair of *weak alternatives*, i.e., two statements at most one of which is true. It also gives additional assumptions under which the statements become strong alternatives. These additional assumptions are called *constraint qualifications*.

Remark 2 Note that if $A_0 = \mathcal{A}(x_0)$ for some x_0, the theorem can be paraphrased as follows: Exactly one of the following statements is true.

1. There exists an $x \in \mathcal{V}$ with $\mathcal{A}(x) \gneqq 0$.
2. There exists a $Z \in \mathcal{S}$ with $Z > 0$ and $\mathcal{A}^{\mathrm{adj}}(Z) = 0$.

If in addition the mapping $\mathcal{A}$ has full rank, i.e., $\mathcal{A}(x) = 0$ implies $x = 0$, then the first statement is equivalent to $\mathcal{A}(x) \geq 0$, $x \neq 0$. $\diamond$

Example 4 Theorem **ALT 2** implies that at most one of the following are possible: either there exists $x \in \mathbf{R}^m$ such that LMI (4) holds, or there exists $Z \in \mathcal{S}$ with $Z > 0$, $Tr\, A_i Z = 0$ for $i = 1, \ldots, m$, and $Tr\, A_0 Z \leq 0$. However, it is possible that neither condition holds; a simple counterexample is provided by $\mathcal{S} = \mathcal{S}^2$, $A_0 = diag\,(0, -1)$ and $A_1 = diag\,(1, 0)$. $\diamond$

Theorem 3 (ALT 3) *At most one of the following statements is true.*

1. *There exists an* $x \in \mathcal{V}$ *with* $\mathcal{A}(x) + A_0 \geq 0$.
2. *There exists a* $Z \in \mathcal{S}$ *with* $Z \geq 0$, $\mathcal{A}^{\mathrm{adj}}(Z) = 0$, *and* $\langle A_0, Z \rangle_{\mathcal{S}} < 0$.

Moreover, if $A_0 = \mathcal{A}(x_0)$ *for some* $x_0 \in \mathcal{V}$, *or if there exists no* $x \in \mathcal{V}$ *such that* $\mathcal{A}(x) \gneqq 0$, *then exactly one of the two statements is true.*

Again, the theorem states a pair of weak alternatives, and additional assumptions under which the statements are strong alternatives.

Note that the theorem is trivial if $A_0 = \mathcal{A}(x_0)$ for some x_0: the first statement is true because we can take $x = -x_0$; the second statement is obviously false because $\mathcal{A}^{\mathrm{adj}}(Z) = 0$ implies that

$$\langle A_0, Z\rangle_{\mathcal{S}} = \langle \mathcal{A}(x_0), Z\rangle_{\mathcal{S}} = \langle x_0, \mathcal{A}^{\mathrm{adj}}(Z)\rangle_{\mathcal{V}} = 0.$$

Example 5 Theorem **ALT 3**, applied to the linear mapping (2), implies that at most one of the following are possible: either there exists $x \in \mathbf{R}^m$ such that LMI (5) holds, or there exists $Z \in \mathcal{S}$ with $Z \geq 0$ such that $\mathrm{Tr}\, A_i Z = 0$, $i = 1, 2, \ldots, m$, and $\mathrm{Tr}\, A_0 Z < 0$. It is possible that neither condition holds; see for example [23, p.378]. ◇

2.2 Semidefinite programming duality

A *semidefinite programming* problem (SDP) requires minimizing a linear function subject to an LMI constraint:

$$\begin{array}{ll} \texttt{minimize:} & \langle c, x\rangle_{\mathcal{V}} \\ \texttt{subject to:} & \mathcal{A}(x) + A_0 \geq 0 \end{array} \tag{8}$$

From convex duality, we can associate with the SDP the *dual* problem

$$\begin{array}{ll} \texttt{maximize:} & -\langle A_0, Z\rangle_{\mathcal{S}} \\ \texttt{subject to:} & \mathcal{A}^{\mathrm{adj}}(Z) = c, \quad Z \geq 0 \end{array} \tag{9}$$

where the variable is the matrix $Z \in \mathcal{S}$. In the context of duality we refer to the SDP (8) as the *primal problem* associated with (9).

The following theorem relates the optimal values of the primal and dual SDPs. Let p_{opt} be the optimal value of (8) and d_{opt} the optimal value of (9). We allow values $\pm\infty$: $p_{\mathrm{opt}} = +\infty$ if the primal problem is infeasible and $p_{\mathrm{opt}} = -\infty$ if it is unbounded below; $d_{\mathrm{opt}} = +\infty$ if the dual problem is unbounded above, $d_{\mathrm{opt}} = -\infty$ if it is infeasible.

Theorem 4 $p_{\mathrm{opt}} \geq d_{\mathrm{opt}}$. *If the primal problem is strictly feasible, (i.e., there exists x with $\mathcal{A}(x) + A_0 > 0$), or the dual problem is strictly feasible (i.e., there exists $Z > 0$ with $\mathcal{A}^{\mathrm{adj}}(Z) = c$), then $p_{\mathrm{opt}} = d_{\mathrm{opt}}$.*

The first property ($p_{\mathrm{opt}} \geq d_{\mathrm{opt}}$) is called *weak duality*. If $p_{\mathrm{opt}} = d_{\mathrm{opt}}$, we say the primal and dual SDPs satisfy *strong duality*. A proof of Theorem 4 can be found in [22].

Theorem 4 is the standard Lagrange duality result for semidefinite programming. An alternative duality theory, which does not require a constraint qualification, was developed by Ramana, Tunçel, and Wolkowicz [38].

2.3 Optimality conditions

Suppose strong duality holds. The following facts are useful when studying the properties of the optimal solutions of the primal and dual SDP.

- A primal feasible x and a dual feasible Z are optimal if and only if $(\mathcal{A}(x)+A_0)Z = 0$. This property is called *complementary slackness*.
- If the primal problem is strictly feasible, then the dual optimum is attained, i.e., there exists a dual optimal Z.
- If the dual problem is strictly feasible, then the primal optimum is attained, i.e., there exists a primal optimal x.

A proof of this result can be found in [22].

We combine these properties to state necessary and sufficient conditions for optimality. For example, it follows that if the primal problem is strictly feasible (hence strong duality obtains), then a primal feasible x is optimal if and only if there exists a dual feasible Z with $(\mathcal{A}(x)+A_0)Z = 0$.

Note that complementary slackness between optimal solutions is only satisfied when strong duality holds; see for example [9, p. 65].

2.4 Some useful preliminaries

We will encounter four specific linear mappings several times in the sequel. For easy reference, we define these here, and derive the expression for their adjoints.

Example 6 Let $\mathcal{A}_1 : \mathcal{S}^n \to \mathcal{S}^n$ be defined by

$$\mathcal{A}_1(P) = -(A^*P + PA).$$

Then, it is easily verified that $\mathcal{A}_1^{\mathrm{adj}} : \mathcal{S}^n \to \mathcal{S}^n$ is given by

$$\mathcal{A}_1^{\mathrm{adj}}(Z) = -(ZA^* + AZ).$$

◇

Example 7 Let $\mathcal{A}_2 : \mathcal{S}^n \to \mathcal{S}^n \times \mathcal{S}^n$ be defined by

$$\mathcal{A}(P) = diag(-(A^*P + PA), P).$$

Then, it is easily verified that $\mathcal{A}_2^{\text{adj}} : \mathcal{S}^n \times \mathcal{S}^n \to \mathcal{S}^n$ is given by

$$\mathcal{A}_2^{\text{adj}}(Z) = -(Z_1 A^* + A Z_1 - Z_2),$$

where $Z = diag(Z_1, Z_2)$. $\diamond$

Example 8 Let $\mathcal{A}_3 : \mathcal{S}^n \to \mathcal{S}^{n+m}$ be defined by

$$\mathcal{A}_3(P) = -\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix}.$$

Then, it is easily verified that $\mathcal{A}_3^{\text{adj}} : \mathcal{S}^{n+m} \to \mathcal{S}^n$ is given by

$$\mathcal{A}_3^{\text{adj}}\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix}\right) = -Z_{11}A^* - AZ_{11} - BZ_{12}^* - Z_{12}B^*.$$

$\diamond$

3 Lyapunov inequalities, stability, and controllability

As our first application of the theorem of alternatives to the analysis of linear time-invariant (LTI) systems, we consider the LTI system

$$\dot{x} = Ax, \tag{10}$$

where $A \in \mathbf{C}^{n \times n}$. Lyapunov equations, i.e., equations of the form $A^*P + PA + Q = 0$, and Lyapunov inequalities, i.e., LMIs of the form $A^*P + PA < 0$ or $A^*P + PA \leq 0$ play a fundamental role in establishing the stability of system (10); see any text on linear systems, for instance, [39].

We consider some well known results on Lyapunov inequalities. Although these results are readily proved using standard techniques, we give a proof using SDP duality to illustrate the techniques that will be used later in the paper.

3.1 Strict Lyapunov inequalities

Proposition 1 *Exactly one of the following two statements is true.*

1. *There exists a* $P \in \mathcal{S}^n$ *such that* $A^*P + PA < 0$.
2. A *has an imaginary eigenvalue.*

Proof. With $\mathcal{A}_1$ as in Example 6 and with $A_0 = 0$, the first statement of the theorem is equivalent to the existence of $P \in \mathcal{S}^n$ such that $\mathcal{A}_1(P) + A_0 > 0$. Then, applying Theorem **ALT 1**, the alternative is that there exists a $Z \in \mathcal{S}^n$ with

$$Z \gneqq 0, \quad AZ + ZA^* = 0. \tag{11}$$

We now show that this condition is equivalent to A having imaginary eigenvalues, establishing the proposition.

Suppose A has an imaginary eigenvalue, i.e., there exist nonzero $v \in \mathbf{C}^n$, and $\omega \in \mathbf{R}$ with $Av = j\omega v$. It is easily shown that $Z = vv^*$ satisfies (11).

Conversely, suppose that (11) holds. Let $Z = UU^*$ where $U \in \mathbf{C}^{n\times r}$ and $Rank\, U = Rank\, Z = r$. From (11), we note that AZ is skew-Hermitian, so that we must have $AUU^* = USU^*$ where S is skew-Hermitian. Therefore $AU = US$. The eigenvalues of S are all on the imaginary axis because S is skew-Hermitian. Therefore, the columns of U span an invariant subspace of A associated with a set of imaginary eigenvalues. Thus A has at least one imaginary eigenvalue. □

Remark 3 In Proposition 1, it is easy to show directly that both statements cannot hold; this is the "easy" part. For instance, if A has an imaginary eigenvalue, i.e., if $Av = j\omega v$ for some $\omega \in \mathbf{R}$ and nonzero $v \in \mathbf{C}^n$, it is easy to show that $A^*P + PA < 0$ cannot hold for any $P \in \mathcal{S}^n$. (In the proof, we prove this "easy" implication with the second alternative.) The hard part is the converse, and the theorems of alternatives give a "constructive" proof: We exhibit the eigenspace of A corresponding to one or more imaginary eigenvalues. It is also worthy of note that (numerical) convex optimization algorithms operate similarly: Given a convex feasibility problem, they either find a feasible point, or provide a constructive proof of infeasibility.

Proposition 1 is representative of most of the results in the sequel, with an easy part and a hard part, with the theorems of alternatives providing a constructive proof of the hard part. ◇

Proposition 2 *Exactly one of the following two statements is true.*

1. *There exists a $P \in \mathcal{S}^n$ such that $P > 0$ and $A^*P + PA < 0$.*
2. *A has an eigenvalue with non-negative real part.*

Remark 4 This is a restatement of the celebrated Lyapunov stability theorem for LTI systems. ◇

Proof. With $\mathcal{A}_2$ as in Example 7 and with $A_0 = 0$, the first statement of the theorem is equivalent to the existence of $P \in \mathcal{S}^n$ such that $\mathcal{A}_2(P) + A_0 > 0$. Then, applying Theorem **ALT 1**, the alternative is that there exist $Z_1 \in \mathcal{S}^n$ and $Z_2) \in \mathcal{S}^n$ with

$$diag(Z_1, Z_2) \gneq 0, \quad Z_1 A^* + A Z_1 - Z_2 = 0. \tag{12}$$

We now show that this condition is equivalent to A having eigenvalues with non-negative real part, establishing the proposition.

Suppose that A has an eigenvalue with non-negative real part, i.e., there exist nonzero $v \in \mathbf{C}^n$, $\sigma \geq 0$ and $\omega \in \mathbf{R}$ with $Av = (\sigma + j\omega)v$. It is easily shown that $Z_1 = vv^*$, $Z_2 = 2\sigma vv^*$ satisfy (12).

Conversely, suppose that (12) holds. We can write $Z_1 = UU^*$ with $U \in \mathbf{C}^{n \times r}$ and $Rank\, U = Rank\, Z = r$. From (12), we note that the symmetric part of AZ_1 is positive semidefinite, so that we must have $AUU^* = USU^*$ where S is the sum of a skew-Hermitian and a positive semidefinite matrix. Then, $AU = US$. The eigenvalues of S are all in the closed right-half plane because S is the sum of a skew-Hermitian and a positive semidefinite matrix. Therefore U spans a (nonempty) invariant subspace of A associated with a set eigenvalues of A with non-negative real part. □

Remark 5 Theorem **ALT 1**, besides offering a simple proof to Lyapunov's theorem, also enables the extension of Proposition 2 to more general settings. Consider the problem of the existence of P satisfying

$$P > 0, \quad A_1^* P + P A_1 < 0, \quad A_2^* P + P A_2 < 0. \tag{13}$$

The matrix P can be interpreted as defining a common or simultaneous quadratic Lyapunov function [40, 1, 41, 42, 43, 44] that proves the stability of the time-varying system

$$\dot{x} = A(t)x,\ A(t) = \lambda(t)A_1 + (1 - \lambda(t))A_2,$$
$$\lambda(t) \in [0, 1] \text{ for all } t.$$

An application of Theorem **ALT 1** immediately yields a necessary and sufficient condition for (13) to be feasible: There do not exist $Z_1, Z_2 \in \mathcal{S}^n$ such that

$$diag(Z_1, Z_2) \gneq 0,\ Z_1 A_1^* + A_1 Z_1 + Z_2 A_2^* + A_2 Z_2 \geq 0. \tag{14}$$

It is easy to show that if $A_1 + \sigma A_2$ has a nonnegative eigenvalue for some $\sigma \in \mathbf{C}$, then (14) is feasible, or there does not exist P satisfying (13). References [41, 42, 43, 44] explore sufficient conditions, using algebraic techniques, for the existence of P satisfying (13) for the special case when the matrices A_i are 2×2 and real.

3.2 Nonstrict Lyapunov inequalities

We saw in §3.1 that the alternatives to strict Lyapunov inequalities involving a matrix A are equivalent to a condition on *some* eigenvalue of A. We will see in this section that the alternatives to nonstrict Lyapunov inequalities result in conditions that are to be satisfied by *all* eigenvalues of A.

Proposition 3 *Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ such that $A^*P + PA \lneqq 0$.*
2. *A is similar to a purely imaginary diagonal matrix.*

Proof. Follows from an application of Theorem **ALT 2**; see [22]. □

Proposition 4 *Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ such that $A^*P + PA \leq 0$, $P \gneqq 0$.*
2. *The eigenvalues of A are in the open right half plane.*

Proof. Follows from an application of Theorem **ALT 2**; see [22]. □

Remark 6 Propositions 1–4 deal with the issue of whether the eigenvalues of A lie in or on the boundary of the left-half complex plane. It is possible to directly extend these propositions to handle general disks in the complex plane (see for example [18]). An indirect route is through conformal mapping techniques from complex analysis (see for instance, [45]). For example, the mapping $A \mapsto (I+A)(I-A)^{-1}$ can be used to derive theorems of alternatives that address whether the eigenvalues of A lie in or on the boundary of the unit disk in the complex plane; the underlying control-theoretic interpretation then concerns the stability of discrete-time linear systems. For a direct extension of Propositions 1–4 to handle to handle general disks, see [22]. ◇

3.3 Lyapunov inequalities with equality constraints

We next consider an LTI system with an input:

$$\dot{x} = Ax + Bu, \tag{15}$$

where $A \in \mathbf{C}^{n\times n}$ and $B \in \mathbf{C}^{n\times m}$. The pair (A,B) is said to be *controllable* if for every initial condition $x(0)$, there exists an input u and T such that $x(T) = 0$. While, there are several equivalent characterizations and conditions for controllability of (A,B) (see for example [39]), we will use the following: The pair (A,B) is not controllable if and only if there exists a left eigenvector v^* of A such that $v^*B = 0$.

If (A,B) is controllable, then given any monic polynomial $a : \mathbf{C} \to \mathbf{C}$ of degree n with complex coefficients, there exists $K \in \mathbf{C}^{m\times n}$ such that $\det(sI - A - BK) = a(s)$ for all $s \in \mathbf{C}$. In other words, with "state-feedback" $u = Kx$ in (15), the eigenvalues of $A + BK$ can be arbitrarily assigned. When (A,B) is not controllable, there exists a nonsingular matrix $T \in \mathbf{C}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{16}$$

where $A_{11} \in \mathbf{C}^{r\times r}$ and $B_1 \in \mathbf{C}^{r\times m}$, with $r < n$ and (A_{11}, B_1) being controllable. (This is called the "Kalman form".) The eigenvalues of A_{22} are called the uncontrollable modes. An uncontrollable mode is called nondefective if its algebraic multiplicity as an eigenvalue of A_{22} equals its geometric multiplicity. The matrix T in (16) has the interpretation of a state coordinate transformation $\bar{x} = T^{-1}x$ such that in the new coordinates, only the first r components of the state are controllable.

Proposition 5 *Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ satisfying $A^*P + PA \lneq 0$, $PB = 0$.*
2. *All uncontrollable modes of (A,B) are nondefective and correspond to imaginary eigenvalues.*

Proof. With $\mathcal{A}_3$ as in Example 8 and with $A_0 = 0$, the first statement of the theorem is equivalent to the existence of $P \in \mathcal{S}^n$ such that $\mathcal{A}_3(P) + A_0 \gneq 0$. Then, applying Theorem **ALT 2**, the alternative is that there exists $Z \in \mathcal{S}^{n+m}$ such that

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} > 0,\ AZ_{11} + Z_{11}A^* + BZ_{12}^* + Z_{12}B^* = 0.$$

Defining $K = Z_{12}^* Z_{11}^{-1}$, we can write this equivalently as

$$Z_{11} > 0, \quad (A+BK)Z_{11} + Z_{11}(A+BK)^* = 0. \tag{17}$$

In other words, the first statement of the Proposition is false if and only if there exist $K \in \mathbf{R}^{n\times m}$ and $Z_{11} \in \mathcal{S}^n$ that satisfy (17). We now establish that this condition is equivalent to the second statement. We will assume, without loss of generality, that (A,B) is in Kalman form, and that K and Z_{11} are appropriately partitioned as

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}, \quad Z_{11} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{12}^* & \tilde{Z}_{22} \end{bmatrix}. \tag{18}$$

Suppose that the uncontrollable modes of (A,B) (if any) are nondefective and correspond to imaginary eigenvalues. We will establish that we can find $Z_{11} > 0$ and K satisfying (17). By assumption A_{22} is similar to a purely imaginary diagonal matrix. The pair (A_{11}, B_1) is controllable, so there exists K_1 such that the eigenvalues of $A_{11} + B_1K_1$ are distinct, purely imaginary, and different from the eigenvalues of A_{22}. Therefore there exist V_{11} and V_{22} such that

$$V_{11}(A_{11} + B_1K_1)V_{11}^{-1} = \Lambda_1, \quad V_{22}A_{22}V_{22}^{-1} = \Lambda_2$$

where Λ_1 and Λ_2 are diagonal and purely imaginary. The spectra of Λ_1 and A_{22} are disjoint, so the Sylvester equation $-\Lambda_1V_{12} + V_{12}A_{22} = -V_{11}A_{12}$ has a unique solution V_{12} (see [37, Th. 4.4.5]). If we take $K_2 = 0$, it is easily verified that $V = \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix}$ satisfies

$$\begin{aligned} &V(A+BK)V^{-1} = \\ &\begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix} \begin{bmatrix} A_{11} + B_1K_1 & A_{12} + B_1K_2 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \end{aligned}$$

i.e., $A + BK$ is similar to a purely imaginary diagonal matrix. We can now proceed as in the proof of Proposition 3 and show that the matrix $Z_{11} = VV^*$ satisfies (17).

Conversely, suppose that Z_{11} and K satisfy (17). In particular, $\tilde{Z}_{22} > 0$, and $A_{22}\tilde{Z}_{22} + \tilde{Z}_{22}A_{22}^* = 0$. As in the proof of Proposition 3 we can construct from $\tilde{Z}_{22}$ a similarity transformation that makes A_{22} diagonal with purely imaginary diagonal elements. Hence all the uncontrollable modes are nondefective and correspond to imaginary eigenvalues. □

Proposition 6 *Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ satisfying $P \gneqq 0$, $A^*P + PA \leq 0$, $PB = 0$.*
2. *All uncontrollable modes of (A,B) correspond to eigenvalues with positive real part.*

Proof. Follows from an application of Theorem **ALT 2**; see [22]. □

Finally, we present a condition for controllability. We first note the following result, which can be interpreted as a theorem of alternatives for linear equations.

Proposition 7 *Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ satisfying*

$$P \neq 0, \quad A^*P + PA = 0, \quad PB = 0 \tag{19}$$

2. *With $\lambda_1, \cdots, \lambda_p$ denoting the uncontrollable modes of (A, B), $\lambda_i + \lambda_j^* \neq 0$, $1 \leq i, j \leq p$.*

Proof. Without loss of generality we can assume that (A, B) is in the Kalman form (16), with $A_{22} \in \mathbf{C}^{p \times p}$. We partition P accordingly as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}.$$

First suppose $\lambda_i + \lambda_j^* = 0$ for two eigenvalues λ_i and λ_j of A_{22}. Then the Lyapunov equation $A_{22}^* P_{22} + P_{22} A_{22} = 0$ has a nonzero solution P_{22} (see [37, Th. 4.4.5]). Taking $P_{11} = 0$ and $P_{12} = 0$, we obtain a nonzero P that satisfies $A^*P + PA = 0$, $PB = 0$.

Conversely, if P satisfies (19), then

$$(A + BK)P + P(A + BK)^* = 0$$

for all K. This is only possible if for all K,

$$A + BK = \begin{bmatrix} A_{11} + B_1 K_1 & A_{22} + B_1 K_2 \\ 0 & A_{22} \end{bmatrix}$$

has eigenvalues μ_i and μ_j that satisfy $\mu_i + \mu_j^* = 0$ (again, see [37, Th. 4.4.5]). The spectrum of $A + BK$ is the union of the spectrum of $A_{11} + B_1 K_1$ and the spectrum of A_{22}. Therefore we must have $\lambda_i + \lambda_j^* = 0$ for two eigenvalues of A_{22}. □

Proposition 8 *Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ satisfying $P \neq 0, \quad A^*P + PA \leq 0, \quad PB = 0$.*
2. *The pair (A, B) is controllable.*

Proof. Statement 1 is true if the statements 1a or 1b listed below are true.

1a. There exists $P \in \mathcal{S}^n$ satisfying $A^*P + PA \lneq 0$, $PB = 0$.

1b. There exists $P \in \mathcal{S}^n$ satisfying $P \neq 0$, $A^*P + PA = 0$, $PB = 0$.

By Propositions 7 and 5 the alternatives to these statements are the following:

2a. All uncontrollable modes are nondefective, and correspond to imaginary eigenvalues.

2b. With $\lambda_1, \cdots \lambda_p$ denoting the uncontrollable modes of (A, B), $\lambda_i + \lambda_j^* \neq 0$, $1 \leq i, j \leq p$.

Thus the alternative to 1 is that 2a *and* 2b are true, i.e., that there are no uncontrollable modes. □

Remark 7 Alternative proofs of this result appeared in [46] and [47, Lemma 1]. ◇

4 Riccati inequalities

We next consider convex Riccati inequalities, which take the form

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M \leq 0, \tag{20}$$

with $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$. Let M be partitioned as $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}$, where $M_{11} = M_{11}^* \in \mathcal{S}^n$. Then, when $M_{22} > 0$, inequality (20) is equivalent to

$$A^*P + PA - M_{11} + (PB - M_{12})M_{22}^{-1}(B^*P - M_{12}^*)^{-1} \leq 0.$$

Such inequalities are widely encountered in quadratic optimal control, estimation theory, and H_∞ control; see for example [48, 49, 50].

4.1 Strict Riccati inequalities

Proposition 9 *Suppose $M_{22} > 0$. Then exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ such that*

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0. \tag{21}$$

2. *For some full-rank $U \in \mathbf{C}^{n\times r}$, $V \in \mathbf{C}^{m\times r}$, and $S \in \mathbf{C}^{r\times r}$ with $S + S^* = 0$,*

$$US - AU = BV, \quad Tr\left(\begin{bmatrix} U^* & V^* \end{bmatrix} M \begin{bmatrix} U \\ V \end{bmatrix}\right) \leq 0.$$

Proof. With $\mathcal{A}_3$ as in Example 8 and with $A_0 = M$, the first statement of the theorem is equivalent to the existence of $P \in \mathcal{S}^n$ such that $\mathcal{A}_3(P) + A_0 > 0$. Then, applying Theorem **ALT 1**, the alternative is that there exists a $Z \in \mathcal{S}^{n+m}$ with

$$\begin{gathered} Z = \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{array} \right] \gneqq 0, \\ Z_{11}A^* + AZ_{11} + Z_{12}B^* + BZ_{12}^* = 0, \quad Tr\, ZM \leq 0 \end{gathered} \tag{22}$$

We now show that this condition is equivalent to the existence of $U \in \mathbf{C}^{n \times r}$, $V \in \mathbf{C}^{m \times r}$, and $S \in \mathbf{C}^{r \times r}$ with $S + S^* = 0$ such that

$$US - AU = BV, \quad Tr \left(\left[\begin{array}{cc} U^* & V^* \end{array} \right] M \left[\begin{array}{c} U \\ V \end{array} \right] \right) \leq 0. \tag{23}$$

We must have $Z_{11} \gneqq 0$, as otherwise we would have $Z_{12} = 0$, and the last inequality in (22) would imply that $Z_{22} = 0$, and consequently $Z = 0$, a contradiction. Therefore, there exist $U \in \mathbf{C}^{n \times r}$ and $V \in \mathbf{C}^{m \times r}$, where $r = Rank\, Z_{11} \geq 1$. such that

$$\left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{array} \right] = \left[\begin{array}{cc} U & 0 \\ V & \hat{V} \end{array} \right] \left[\begin{array}{cc} U^* & V^* \\ 0 & \hat{V}^* \end{array} \right]$$

where U has full rank. The equation $Z_{11}A^* + AZ_{11} + Z_{12}B^* + BZ_{12}^* = 0$, represented in terms of U and V means that $AUU^* + BVU^*$ is skew-Hermitian, i.e., it can be written as $AUU^* + BVU^* = USU^*$, where S is skew-Hermitian. Since U has full rank, this last equation implies $AU + BV = US$. Expressing the inequality $Tr\, ZM \leq 0$ in terms of U and V, we obtain

$$Tr \left(\left[\begin{array}{cc} U^* & V^* \\ 0 & \hat{V}^* \end{array} \right] M \left[\begin{array}{cc} U & 0 \\ V & \hat{V} \end{array} \right] \right) \leq 0,$$

which, since $M_{22} > 0$, implies that

$$Tr \left(\left[\begin{array}{cc} U^* & V^* \end{array} \right] M \left[\begin{array}{c} U \\ V \end{array} \right] \right) \leq 0,$$

completing the proof. □

The conclusion of Proposition 9 can be further developed to yield the Kalman-Yakubovich-Popov Lemma.

Lemma 1 (KYP Lemma) *Suppose $M_{22} > 0$. There exists $P \in \mathcal{S}^n$ such that*

$$\left[\begin{array}{cc} A^*P + PA & PB \\ B^*P & 0 \end{array} \right] - M < 0, \tag{24}$$

if and only for all $\omega \in \mathbf{R}$,

$$(j\omega I - A)u = Bv, \ (u, v) \neq 0 \Longrightarrow \left[\begin{array}{cc} u^* & v^* \end{array} \right] M \left[\begin{array}{c} u \\ v \end{array} \right] > 0. \tag{25}$$

Proof. Suppose that there does not exist $P \in \mathcal{S}^n$ such that (24) holds. From Proposition 9, this is equivalent to the existence of a full-rank $U \in \mathbf{C}^{n\times r}$, $V \in \mathbf{C}^{m\times r}$, and $S \in \mathbf{C}^{r\times r}$ with $S + S^* = 0$, such that

$$US - AU = BV, \quad Tr\left(\left[\begin{array}{cc} U^* & V^* \end{array}\right] M \left[\begin{array}{c} U \\ V \end{array}\right]\right) \leq 0. \tag{26}$$

We show that (26) is equivalent to the existence of $u \in \mathbf{C}^n$ and $v \in \mathbf{C}^m$, not both zero, such that (25) does not hold at some ω.

Suppose there exist $u \in \mathbf{C}^n$ and $v \in \mathbf{C}^m$, not both zero, such that (25) does not hold at some ω. Then, it is easy to verify that (26) holds with

$$U = [\Re u \;\; \Im u], \quad V = [\Re v \;\; \Im v], \quad S = \left[\begin{array}{cc} 0 & -\omega \\ \omega & 0 \end{array}\right].$$

Conversely suppose that there exist full-rank $U \in \mathbf{C}^{n\times r}$, $V \in \mathbf{C}^{m\times r}$, and $S \in \mathbf{C}^{r\times r}$ with $S + S^* = 0$, such that (26) holds. We then take the Schur decomposition of S: $S = \sum_{i=1}^m j\omega_i q_i q_i^*$, where $\sum_i q_i q_i^* = I$. We then have

$$\begin{aligned} 0 &\geq Tr\left(\left[\begin{array}{c} U \\ V \end{array}\right]^* M \left[\begin{array}{c} U \\ V \end{array}\right]\right) \\ &= Tr\left(\left[\begin{array}{c} U \\ V \end{array}\right]^* M \left[\begin{array}{c} U \\ V \end{array}\right] \sum_i q_i q_i^*\right) \\ &= \sum_{i=1}^m q_i^* \left[\begin{array}{c} U \\ V \end{array}\right]^* M \left[\begin{array}{c} U \\ V \end{array}\right] q_i. \end{aligned}$$

At least one of the m terms in this last expression must be less than or equal to zero. Let k be the index of that term, and define $u = Uq_k$, $v = Vq_k$. (u is nonzero because U has full rank.) We have

$$\left[\begin{array}{cc} u^* & v^* \end{array}\right] M \left[\begin{array}{c} u \\ v \end{array}\right] \leq 0$$

and, by multiplying $US - AU = BV$ with q_k on the right, $Au + Bv = j\omega_k u$. In other words we have constructed a u and v showing that (25) does not hold at $\omega = \omega_k$. □

Remark 8 Our statement of the KYP Lemma is more general than standard versions (see for example, [16]), as we allow A to have imaginary eigenvalues. If A has no imaginary eigenvalues, then (25) simply means that

$$\left[\begin{array}{cc} B^*(-j\omega I - A^*)^{-1} & I \end{array}\right] M \left[\begin{array}{c} (j\omega I - A)^{-1}B \\ I \end{array}\right] > 0. \tag{27}$$

The following form of the frequency-domain condition is more commonly found in the literature: the inequality (27) holds for all ω where $j\omega I - A$ is invertible. If A has imaginary eigenvalues, then this condition is weaker than requiring that (25) holds for all ω, and it is not equivalent to feasibility of the LMI (21). See [22] for a counterexample. $\diamond$

We next use the theorem of alternatives to exhibit the well-known connection between the KYP lemma and a certain Hamiltonian matrix.

Proposition 10 *Suppose that A has no imaginary eigenvalues and that $M_{22} > 0$. Then, exactly one of the following statements is true.*

1. *There exists $P \in \mathcal{S}^n$ such that (21) holds.*
2. *The Hamiltonian matrix*

$$H = \begin{bmatrix} A - BM_{22}^{-1}M_{12}^* & BM_{22}^{-1}B^* \\ M_{11} - M_{12}M_{22}^{-1}M_{12}^* & -(A - BM_{22}^{-1}M_{12}^*)^* \end{bmatrix}$$

   has an imaginary eigenvalue.

Proof. We established in the proof of Proposition 9 that the condition that there does not exist $P \in \mathcal{S}^n$ such that (21) holds is equivalent to the existence of $Z \in \mathcal{S}^{n+m}$ such that (22) holds. It can be shown (see [22]) that this condition is equivalent to H having imaginary eigenvalues. $\square$

4.2 Strict Riccati inequality with positive definite P

Proposition 11 *Suppose $M_{22} > 0$. Exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ such that*

$$P > 0, \quad \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0.$$

2. *For some full-rank $U \in \mathbf{C}^{n \times r}$, $V \in \mathbf{C}^{m \times r}$, and $S \in \mathbf{C}^{r \times r}$ with $S + S^* \geq 0$,*

$$US - AU = BV, \quad \mathrm{Tr}\left(\begin{bmatrix} U^* & V^* \end{bmatrix} M \begin{bmatrix} U \\ V \end{bmatrix} \right) \leq 0. \tag{28}$$

Proof. Follows from an application of Theorem **ALT 1**; see [22]. $\square$

Frequency-domain interpretation. Recall that we were able to extend Proposition 9 to yield the KYP Lemma, which establishes the connection between an LMI and a certain frequency-domain condition. Unfortunately, as far we know, no such extensions are possible in general with Proposition 11. However, when M satisfies additional constraints, it is possible to provide a frequency-domain interpretation for Proposition 11.

Proposition 12 *Suppose $M_{22} > 0$, $M_{11} \leq 0$, and all the eigenvalues of A have negative real part. There exists $P \in \mathcal{S}^n$ such that*

$$P > 0, \quad \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0 \tag{29}$$

if and only if for all $s \in \mathbf{C}$ with $\Re s \geq 0$,

$$\begin{bmatrix} B^*(sI - A)^{-*} & I \end{bmatrix} M \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} > 0. \tag{30}$$

Proof. See [22].

4.3 Nonstrict Riccati inequalities

Proposition 13 *Suppose $M_{22} \geq 0$ and that all uncontrollable modes of (A, B) are nondefective and correspond to imaginary eigenvalues. Then, exactly one of the following two statements is true.*

1. *There exists $P \in \mathcal{S}^n$ such that*

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M \leq 0. \tag{31}$$

2. *For some full-rank $U \in \mathbf{C}^{n \times n}$, $V \in \mathbf{C}^{m \times r}$, and $S \in \mathbf{C}^{r \times r}$ with $S + S^* = 0$,*

$$US - AU = BV, \quad \mathrm{Tr}\left(\begin{bmatrix} U^* & V^* \end{bmatrix} M \begin{bmatrix} U \\ V \end{bmatrix} \right) < 0.$$

Proof. Follows from an application of Theorem **ALT 3** and Proposition 5; see [22]. □

Remark 9 The conclusions of Proposition 13 are closely related to conditions for the solvability of Algebraic Riccati Equations (AREs) and Inequalities (ARIs), derived by Scherer [51, 52], for systems with uncontrollable modes on the imaginary axis. Scherer's approach is to reduce the original problem to that of solvability of an ARE for a smaller controllable system, with auxiliary LMIs of the form $A^*P + PA + S \geq 0$ where A has purely imaginary eigenvalues, and P is required to be "arbitrarily large". ◇

As with Proposition 9, the conclusion of Proposition 13 can be further developed to yield the nonstrict version of the Kalman-Yakubovich-Popov Lemma.

Lemma 2 (KYP Lemma, nonstrict version) *Suppose that $M_{22} \geq 0$ and that all uncontrollable modes of (A,B) are nondefective and correspond to imaginary eigenvalues. There exists $P \in \mathcal{S}^n$ such that*

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M \leq 0,$$

if and only if for all $\omega \in \mathbf{R}$,

$$(j\omega I - A)u = Bv, \ (u,v) \neq 0 \Longrightarrow \begin{bmatrix} u^* & v^* \end{bmatrix} M \begin{bmatrix} u \\ v \end{bmatrix} \geq 0.$$

Next, we have another variation of Proposition 13, where we impose constraints on P.

Proposition 14 *Suppose $M_{22} \geq 0$ and that all uncontrollable modes of (A,B) correspond to eigenvalues with positive real part. Then, exactly one of the following two statements is true.*

1. *Then there exists $P \in \mathcal{S}^n$ such that*

$$P \geq 0, \quad \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M \leq 0. \tag{32}$$

2. *For some full-rank $U \in \mathbf{C}^{n\times n}$, $V \in \mathbf{C}^{m\times r}$, and $S \in \mathbf{C}^{r\times r}$ with $S + S^* \geq 0$,*

$$US - AU = BV, \quad Tr\left(\begin{bmatrix} U^* & V^* \end{bmatrix} M \begin{bmatrix} U \\ V \end{bmatrix}\right) < 0. \tag{33}$$

Proof. Follows from an application of Theorem **ALT 3** and Proposition 6; see [22]. □

5 The linear quadratic regulator problem

In §4, we considered convex Riccati inequalities, and explored system-theoretic interpretations of conditions for their feasibility via the theorems of alternatives. In this section, we consider the Linear Quadratic Regulator (LQR) problem, which is a classical semidefinite program with convex Riccati inequalities.

Consider the semidefinite program

$$\begin{array}{ll} \texttt{maximize:} & x_0^* P x_0 \\ \texttt{subject to:} & \begin{bmatrix} A^*P + PA + Q & PB \\ B^*P & I \end{bmatrix} \geq 0, \ P \geq 0, \end{array} \tag{34}$$

with $Q \geq 0$. It can be shown (see [22]) that the dual problem of (34) is

$$\begin{array}{ll} \texttt{maximize:} & -Tr\, QZ_{11} - Tr\, Z_{22} \\ \texttt{subject to:} & AZ_{11} + BZ_{12}^* + Z_{11}A^* + Z_{12}B^* + x_0x_0^* \leq 0, \\ & \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \geq 0, \end{array} \tag{35}$$

with variables $Z_{11} \in \mathcal{S}^n$, $Z_{12} \in \mathbf{C}^{n\times m}$, $Z_{22} \in \mathcal{S}^m$.

Interpretation of the primal problem. Consider the following optimal control problem: For the system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{36}$$

$$\begin{array}{l} \text{find } u \in \mathbf{L}_{2e} \text{ that minimizes} \\ J = \int_0^\infty (x(t)^* Q x(t) + u(t)^* u(t))\, dt, \end{array} \tag{37}$$

with $Q \geq 0$, subject to $\lim_{t\to\infty} x(t) = 0$. Let J_{opt} denote the minimum value.

We can write down a lower bound for J_{opt} using quadratic functions. Suppose for $P \geq 0$ we have

$$\frac{d}{dt} x(t)^* P x(t) \geq -\left(x(t)^* Q x(t) + u(t)^* u(t)\right), \tag{38}$$

for all $t \geq 0$, and for all x and u satisfying $\dot{x} = Ax + Bu$, $x(T) = 0$. Then, integrating both sides from 0 to T, we get

$$x_0^* P x_0 \leq \int_0^T (x(t)^* Q x(t) + u(t)^* u(t))\, dt,$$

or we have a lower bound for J_{opt}.

Condition (38) holds for *all* x and u (not necessarily those that steer state to zero) if the LMI

$$\begin{bmatrix} A^*P + PA + Q & PB \\ B^*P & I \end{bmatrix} \geq 0$$

is satisfied. Thus, the optimal value of the SDP (34) provides a lower bound to the optimal value of Problem (37).

Interpretation of the dual problem. Consider system (36) with a constant, linear state-feedback $u = Kx$ such that the state trajectory of the feedback system $\dot{x} = (A + BK)x$, $x(0) = x_0$, satisfies $x(t) \to 0$ as $t \to \infty$. Then the LQR objective J reduces to

$$J_K = \int_0^\infty x(t)^* (Q + K^*K)\, x(t)\, dt.$$

Clearly, for every K, J_K yields an upper bound on the optimum LQR objective J_{opt}.

It can be shown using standard techniques from control theory that the condition that the solution x of $\dot{x} = (A + BK)x$, $x(0) = x_0$, satisfies $x(t) \to 0$ as $t \to \infty$ is equivalent to the feasibility of the conditions

$$(A + BK)\tilde{Z} + \tilde{Z}(A + BK)^* + x_0 x_0^* = 0, \quad \tilde{Z} \geq 0.$$

Moreover, $J_K = Tr\,\tilde{Z}(Q+K^*K)$. Thus, the best upper bound on J_{opt}, achievable using state-feedback control, is given by the optimization problem with the optimization variables $\tilde{Z}$ and K:

$$\begin{array}{ll} \text{minimize:} & Tr\,\tilde{Z}(Q+K^*K) \\ \texttt{subject to:} & \tilde{Z} \geq 0 \\ & (A+BK)\tilde{Z}+\tilde{Z}(A+BK)^*+x_0x_0^* = 0, \end{array}$$

which has the same objective value as (35) evaluated at $Z_{11} = \tilde{Z}$, $Z_{12} = \tilde{Z}K^*$, $Z_{22} = K\tilde{Z}K^*$.

Condition for strict primal feasibility. From Proposition 11, strict primal feasibility is equivalent to the condition that there does not exist a full-rank $U \in \mathbf{C}^{n\times r}$, $V \in \mathbf{C}^{m\times r}$, and $S \in \mathbf{C}^{r\times r}$ with $S+S^* \geq 0$, such that

$$US - AU = BV, \quad Tr\,U^*QU + V^*V \leq 0. \tag{39}$$

As $Q \geq 0$, condition (39) is equivalent to $QU = 0$ and $V = 0$, or we have $AU = US$, $\quad QU = 0$, which is equivalent to (Q,A) having unobservable modes in the closed-right half complex plane [39]. In other words, strict primal feasibility is equivalent to (Q,A) having no unobservable modes corresponding to eigenvalues with nonnegative real part.

Condition for strict dual feasibility. Suppose the dual problem is strictly feasible, that is, there exist $Z_{11} \in \mathcal{S}^n$ and $Z_{12} \in \mathbf{C}^{n\times m}$ such that $Z_{11} > 0$ and $AZ_{11} + BZ_{12}^* + Z_{11}A^* + Z_{12}B^* + x_0x_0^* < 0$. With $K = Z_{12}^*Z_{11}^{-1}$, we then have

$$Z_{11} > 0, \quad (A+BK)Z_{11} + Z_{11}(A+BK) < 0,$$

or (A,B) is stabilizable, that is, all the uncontrollable modes are in the open left-half complex plane [39].

Optimality conditions. If primal and dual are strictly feasible, then strong duality holds, and primal and dual optima are attained. By complementary slackness, we have

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \begin{bmatrix} A^*P+PA+Q & PB \\ B^*P & I \end{bmatrix} = 0,$$

i.e.,

$$\begin{bmatrix} I \\ K \end{bmatrix} \begin{bmatrix} I & K^* \end{bmatrix} \begin{bmatrix} A^*P+PA+Q & PB \\ B^*P & I \end{bmatrix} = 0,$$

or

$$\begin{bmatrix} I & K^* \end{bmatrix} \begin{bmatrix} A^*P+PA+Q & PB \\ B^*P & I \end{bmatrix} = 0,$$

or $K = -B^*P$, with all the eigenvalues of $A+BK$ having negative real part, and

$$A^*P + PA + Q - PBB^*P = 0. \tag{40}$$

This is the classical LQR result, which states that when (A,B) is stabilizable and (Q,A) is detectable, then the optimal control u that solves Problem (37) is a constant state-feedback, with the feedback gain given via the stabilizing solution to the Algebraic Riccati Equation (40).

6 SDP duality and bounds on the H_∞-norm

Consider the LTI system

$$\dot{x} = Ax + Bu, \ \ x(0) = 0, \quad y = Cx, \tag{41}$$

where $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$, and $C \in \mathbf{C}^{p\times n}$, with all the eigenvalues of A having a negative real part. Let (A,B,C) be a minimal realization, and let H denote the transfer function, i.e., $H(s) = C(sI-A)^{-1}B$.

The $\mathbf{H}_\infty$ norm of H is defined as

$$\|H\|_\infty = \sup_{\Re s>0} \sigma_{\max}(H(s)),$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value. It turns out that we also have

$$\begin{aligned} \|H\|_\infty &= \sup_{\omega\in\mathbf{R}} \sigma_{\max}(H(j\omega)) && (42)\\ &= \left(\sup_{u,T_1,T_2} \left\{ \int_{T_1}^{T_2} y(t)^* y(t)\, dt \;\middle|\; \int_{T_1}^{T_2} u(t)^* u(t)\, dt \le 1 \right\} \right)^{1/2}. && (43) \end{aligned}$$

Equality (43) means that $\|H\|_\infty$ is the $\mathbf{L}_2$ gain of system (41), and equality (42) means that $\|H\|_\infty$ is the $\mathbf{L}_2$ gain of system (41) over all possible sinusoidal inputs, i.e., it is the $\mathbf{L}_2$-gain of system (41) over all frequencies.

It is well-known (see for example [1]) that the the optimal value of the SDP

$$\begin{array}{ll} \texttt{minimize:} & \beta \\ \texttt{subject to:} & \begin{bmatrix} A^*P + PA + C^*C & PB \\ B^*P & -\beta I \end{bmatrix} \le 0 \end{array} \tag{44}$$

in the variables $P \in \mathcal{S}^n$ and $\beta \in \mathbf{R}$ is equal to $\|H\|_\infty^2$. It can be shown (see [22]) that the dual problem of (44) is

$$\begin{array}{ll} \texttt{maximize:} & Tr\, CZ_{11}C^* \\ \texttt{subject to:} & Z_{11}A^* + AZ_{11} + Z_{12}B^* + BZ_{12}^* = 0, \\ & \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \ge 0, \quad Tr\, Z_{22} = 1, \end{array} \tag{45}$$

with variables $Z_{11} \in \mathcal{S}^n$, $Z_{12} \in \mathbf{C}^{n\times m}$, $Z_{22} \in \mathcal{S}^m$.

Control-theoretic interpretations of the lower bound

Any feasible point to Problem (45) yields a lower bound on $\|H\|_\infty^2$. We now provide control-theoretic interpretations of such a lower bound.

Time-domain interpretation. Let $u(t)$ be any input that steers the state of system (41) from $x(T_1) = 0$ to $x(T_2) = 0$ for some $T_1, T_2 \in \mathbf{R}$, with $\int_{T_1}^{T_2} u(t)^* u(t)\, dt = 1$. Let $y(t)$ be the corresponding output. Then, from (43), the quantity $\int_{T_1}^{T_2} y(t)^* y(t)\, dt$ serves as a lower bound to $\|H\|_\infty^2$. Define

$$\begin{aligned} Z_{11} &= \textstyle\int_{T_1}^{T_2} x(t)x(t)^*\, dt, \\ Z_{12} &= \textstyle\int_{T_1}^{T_2} x(t)u(t)^*\, dt, \\ Z_{22} &= \textstyle\int_{T_1}^{T_2} u(t)u(t)^*\, dt. \end{aligned}$$

It can be shown (see [22]) that Z_{11}, Z_{12} and Z_{22} are dual feasible. The dual objective value is $\mathrm{Tr}\, C Z_{11} C^* = \int_{T_1}^{T_2} y(t)^* y(t)\, dt$, completing the connection between the control-theoretic interpretation (43), and the dual problem (45).

Frequency-domain interpretation. Let $\omega \in \mathbf{R}$, and let $U \in \mathbf{C}^m$ with $U^*U = 1$. Define $X = (j\omega I - A)^{-1} BU$, $Z_{11} = \Re X X^*$, $Z_{12} = \Re X U^*$, and $Z_{22} = \Re U U^*$. It can be shown (see [22]) that Z_{11}, Z_{12} and Z_{22} are dual feasible. The value of the dual objective function is

$$\begin{aligned} \mathrm{Tr}\, C^* C Z_{11} &= X^* C^* C X \\ &= U^* B^* (-j\omega I - A^*)^{-1} C^* C (j\omega I - A) BU, \end{aligned} \tag{46}$$

which, from (42), is a lower bound on $\|H\|_\infty^2$. The control-theoretic interpretation of the above development follows immediately from (42): As $U^*U = 1$, (46) implies that $\sigma_{\max}(H(j\omega)) \geq \sqrt{\mathrm{Tr}\, C^* C Z_{11}}$, and consequently that $\|H\|_\infty \geq \sqrt{\mathrm{Tr}\, C^* C Z_{11}}$.

Relation to Enns-Glover lower bound

Let W_c and W_o be the controllability and observability Gramians of the system (41) respectively, that is, $AW_c + W_c A^* + BB^* = 0$, and $W_o A + A^* W_o + C^* C = 0$. Let z be a unit-norm eigenvector corresponding to the largest eigenvalue of $W_c^{1/2} W_o W_c^{1/2}$, and let X and Y be the solutions of the two Lyapunov equations

$$\begin{aligned} AY + YA^* + W_c^{1/2} z z^* W_c^{1/2} &= 0, \\ A^* X + XA + W_c^{-1/2} z z^* W_c^{-1/2} &= 0. \end{aligned}$$

Define Z as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} = \begin{bmatrix} Y + W_c X W_c & W_c X B \\ B^* X W_c & B^* X B \end{bmatrix}.$$

It can be verified (see [22]) that Z is dual feasible. Moreover the dual objective value is

$$Tr\, C Z_{11} C^* = Tr\, C Y C^* + Tr\, C W_c X W_c C^* \geq Tr\, C Y C^* = \bar{\sigma}$$

where $\bar{\sigma}$ is the largest eigenvalue of $W_c W_o$. This lower bound on $\|H\|_\infty^2$ is the well-known Enns-Glover lower bound [53, 54]. Note that the duality-based bound, $Tr\, C Y C^* + Tr\, C W_c X W_c C^*$, is *guaranteed* to be at least as good as the Enns Glover bound.

New duality-based upper and lower bounds

Noting that every primal feasible point gives an upper bound and every dual feasible point gives a lower bound, it is possible to generate new bounds for $\|H\|_\infty$. It is readily checked that these bounds are often better than existing bounds.

New upper bounds. It is easily checked that $(2W_o, 4\lambda_{\max}(W_o B B^* W_o, C^* C)) \in \mathcal{S}^n \times \mathbf{R}$ is a primal feasible point, where $\lambda_{\max}(R, S)$ is the maximum generalized eigenvalue of (R, S). Therefore one upper bound on $\|H\|_\infty$ is given by $2\sqrt{\lambda_{\max}(W_o B B^* W_o, C^* C)}$.

Let $\tilde{H}$ be defined by $\tilde{H}(s) = H(s)^T$; then we have $\|H\|_\infty = \|\tilde{H}\|_\infty$, which yields another upper found for $\|H\|_\infty$: $2\sqrt{\lambda_{\max}(W_c C^* C W_c, B B^*)}$.

New lower bounds. It is easily verified that $Z_{11} = W_c/\alpha$, $Z_{12} = B/(2\alpha)$, $Z_{22} = B^* W_c^{-1} B/(4\alpha)$, where $\alpha = Tr\,(B^* W_c^{-1} B/4)$, are dual feasible. Therefore a lower bound on $\|H\|_\infty$ is given by $2\sqrt{Tr\, C W_c C^* / (Tr\, B^* W_c^{-1} B)}$.

Once again noting $\|H\|_\infty = \|\tilde{H}\|_\infty$, where $\tilde{H}(s) = H(s)^T$, we have another lower bound $\|H\|_\infty$:

$$2\sqrt{Tr\, B^* W_o B / (Tr\, C W_o^{-1} C^*)}.$$

7 Conclusions

We have explored the application of semidefinite programming duality in order to obtain new insight, as well as to provide new and simple proofs for some classical results for linear time-invariant systems. We have also shown how SDP duality can be used to derive new results, such as new LMI criteria for controllability (and observability) properties, as well as new upper and lower bounds for the $\mathbf{H}_\infty$ norm.

Duality theory also holds promise in devising more efficient numerical optimization algorithms for certain classes of SDPs derived from the KYP lemma. For recent work in this direction, see [55, 56, 57].

Finally, we note that duality theory and semidefinite programming underlie some of the most promising methods for the study of nonconvex polynomial optimization problems. There is considerable research effort along these directions; see for example [58, 59, 60].

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