INTERIOR-POINT METHODS FOR MAGNITUDE FILTER DESIGN

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ABSTRACT

We describe efficient interior-point methods for the design of filters with constraints on the magnitude spectrum, for example, piecewise-constant upper and lower bounds, and arbitrary phase. Several researchers have observed that problems of this type can be solved via convex optimization and spectral factorization. The associated optimization problems are usually solved via linear programming or, more recently, semidefinite programming. The semidefinite programming approach is more accurate but also more expensive, because it requires the introduction of a large number of auxiliary variables. In this paper we propose a more efficient method, based on convex optimization duality, and on interior-point methods for problems with generalized inequalities.

1. INTRODUCTION

Finite impulse response (FIR) filter design problems often include magnitude constraints of the form

$$L \le |H(e^{j\omega})| \le U, \quad \omega \in [\alpha, \beta] \tag{1}$$

where $0 \le \alpha < \beta \le \pi$, $U > L \ge 0$, and H is the filter transfer function defined as

$$H(z) = h_0 + h_1 z^{-1} + \cdots + h_n z^{-n}.$$

Magnitude constraints are not convex in the filter coefficients $h_0, \dots, h_n$. However, it has been pointed out in [1, 2, 3, 4, 5, 6] that the constraints are convex if we use the *autocorrelation coefficients* x_k, defined as

$$x_k = \sum_{i=0}^{n-k} h_i h_{k+i}, \quad k = 0, \dots, n, \tag{2}$$

as variables. The Fourier transform of the autocorrelation coefficients (assuming $x_k = 0$ for $|k| > n$ and $x_{-k} = x_k$) is given by

$$X(e^{j\omega}) = x_0 + 2\sum_{k=1}^{n} x_k \cos k\omega.$$

We have $X(e^{j\omega}) = |H(e^{j\omega})|^2$, and therefore the constraints (1) reduce to

$$L^2 \le x_0 + 2\sum_{k=1}^{n} x_k \cos k\omega \le U^2, \quad \omega \in [\alpha, \beta] \tag{3}$$

when written in terms of the autocorrelation coefficients x_k. This is an infinite set of linear inequalities in the coefficients x_k (two linear inequalities for each ω), hence a convex constraint. Moreover it is well known that a vector $x \in \mathbf{R}^{n+1}$ can be expressed as (2) if and only if

$$X(e^{j\omega}) \ge 0, \quad \omega \in [0, \pi], \tag{4}$$

which again is a convex constraint in the autocorrelation coefficients x. In summary, in a FIR filter design problem where all the constraints are of the form (1), we can replace the nonconvex constraints (1) by convex constraints (3) and (4), solve for the optimal values of the autocorrelation coefficients, and then find the filter coefficients h by spectral factorization.

As an example, consider the problem of designing a multiband FIR filter with N bands $[\alpha_k, \beta_k]$, $k = 1, 2, \dots, N$. (We assume that $0 \le \alpha_k < \beta_k \le \pi$, and that none of the intervals overlap.) In each band, we have a lower bound $L_k \ge 0$ and an upper bound $U_k > L_k$ on the filter magnitude. We are interesting in minimizing the stopband squared error subject to peak constraints on the magnitude response [7]. This design problem can be expressed as

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{N} w_k \int_{\alpha_k}^{\beta_k} \left|H(e^{j\omega})\right|^2 d\omega \\ \text{subject to} & L_k \le \left|H(e^{j\omega})\right| \le U_k, \\ & \quad \omega \in [\alpha_k, \beta_k], \; k = 1, \dots, N, \end{array} \tag{5}$$

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where $w_k = 0$ if band k is a passband, and $w_k = 1$ if band k is a stopband. (Taking different positive weights w_k allows us to balance the minimization over different stopbands.) In terms of the autocorrelation coefficients x the problem reduces to

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{N} w_k \int_{\alpha_k}^{\beta_k} X(e^{j\omega}) d\omega \\ \text{subject to} & L_k^2 \leq X(e^{j\omega}) \leq U_k^2, \\ & \quad \omega \in [\alpha_k, \beta_k], \; k = 1, \ldots, N \\ & X(e^{j\omega}) \geq 0, \; \omega \in [0, \pi], \end{array} \tag{6}$$

which is a convex problem in the variables x: the objective function is linear in x; the constraints are an infinite set of linear inequalities. From the optimal solution x we can obtain filter coefficients h via spectral factorization.

Two methods exist for dealing with the semi-infinite nature of the constraint (3). The most popular method is to sample the frequency response, *i.e.*, we replace the constraint (3) with a large finite set of inequality constraints Using this approach, we can approximate problem (6) as a linear program with a large, but finite, set of inequalities.

A second and more recent method is based on semidefinite programming. When $\alpha = 0$ and $\beta = \pi$ it is well known that the constraint (3) can be cast as two linear matrix inequality (LMIs), via the positive real lemma [2]. Davidson *et al.* in [5] have recently extended this formulation, and derived an LMI formulation of the constraints (3) for *arbitrary* α and β. Using their method, a problem such as (6) can be cast as a semidefinite programming problem (SDP), without any approximation or sampling, and solved via general-purpose semidefinite programming software. The drawback of this method is that it requires a number of auxiliary variables that often far exceeds the number of variables x.

Our approach in this paper is similar but more efficient than the LMI approach. It is based on a new formulation of the magnitude constraints as generalized linear inequalities, and does not require any auxiliary variables. A related method recently appeared in [8].

2. CONE PROGRAM FORMULATION

We say $x \in \mathbf{R}^{n+1}$ a *finite autocorrelation sequence* if it can be expressed as (2) for some $h \in \mathbf{R}^{n+1}$. Autocorrelation sequences can be characterized in the frequency domain via (4). This is an infinite set of linear inequalities in x (one at each frequency), and therefore the autocorrelation sequences in $\mathbf{R}^{n+1}$ form a closed convex cone. This justifies the following notation. We write $x \succeq 0$ to denote that x is a finite autocorrelation sequence, and more generally, $x \succeq y$ means that $x - y$ is a finite autocorrelation sequence. With this notation, we can express the constraint (4) as a 'generalized' linear inequality $x \succeq 0$. As we will now show, the upper and lower bounds (3) can also be expressed as generalized linear inequalities. This observation allows us to express a wide variety of magnitude filter design problems as linear programming problems with generalized inequalities (also known as *cone programs*), *i.e.*, problems of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_k x + g_k \preceq 0, \quad k = 1, \ldots, L, \end{array} \tag{7}$$

where $c \in \mathbf{R}^{n+1}$, $F_k \in \mathbf{R}^{n_k \times (n+1)}$, and $g_k \in \mathbf{R}^{n_k}$.

We first consider the constraint

$$X(e^{j\omega}) = x_0 + 2\sum_{k=1}^{n} x_k \cos k\omega \geq 0, \;\; \omega \in [\alpha, \beta]. \tag{8}$$

To simplify notation, we make a change of variable $t = \cos\omega$. This maps the interval $0 \leq \omega \leq \pi$ to $-1 \leq t \leq 1$, and the function $\cos k\omega$ to the kth *Chebyshev polynomial* $p_k(t) = \cos(k \cos^{-1} t)$ [9, p.684]. Therefore $X(e^{j\omega})$ is mapped to the polynomial

$$P(t) = x_0 p_0(t) + 2\sum_{k=1}^{n} x_k p_k(t),$$

so it is clear that $P(t) \geq 0$ for $t \in [-1, 1]$ if and only if $x \succeq 0$. Now consider the constraint (8). It is satisfied if and only if $P(t) \geq 0$ for $t \in [\cos\beta, \cos\alpha]$. Let $A(\alpha, \beta) \in \mathbf{R}^{(n+1)\times(n+1)}$ be defined as follows: the components of $A(\alpha, \beta)x$ are the coordinates of P in the basis $p_0(at - b)$, $2p_1(at-b), \ldots, 2p_n(at-b)$, where $a = 2/(\cos\beta - \cos\alpha)$, and $b = (\cos\beta + \cos\alpha)/(\cos\beta - \cos\alpha)$, *i.e.*, if we take $y = A(\alpha, \beta)x$, we can express $P(t)$ as

$$P(t) = y_0\, p_0(at - b) + 2\sum_{k=1}^{n} y_k\, p_k(at - b). \tag{9}$$

(See [10] for the details of constructing the matrix $A(\alpha, \beta)$ for given α, β.) From (9) and the definition of a and b, it is clear that $P(t) \geq 0$ for $t \in [\cos\beta, \cos\alpha]$ if and only if

$$y_0\, p_0(\tau) + 2\sum_{k=1}^{n} y_k\, p_k(\tau) \geq 0, \quad \tau \in [-1, 1].$$

In other words, x satisfies the constraint (8) if and only if $A(\alpha, \beta)x \succeq 0$. More generally, we can express the magnitude constraints (3) as a pair of generalized linear inequalities

$$L^2 e \preceq A(\alpha, \beta)x \preceq U^2 e$$

where $e = (1, 0, \ldots, 0)$ is the first unit vector in $\mathbf{R}^{n+1}$.

Returning to the example of §1, we can use the generalized inequality notation to express problem (6) as the following cone program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & L_k^2 e \preceq A_k x \preceq U_k^2 e, \quad k = 1, \ldots, N \\ & x \succeq 0 \end{array} \qquad (10)$$

where $A_k = A(\alpha_k, \beta_k)$ and

$$c_0 = \sum_{k=1}^{N} w_k(\beta_k - \alpha_k), \quad c_k = 2 \sum_{k=1}^{N} w_k \int_{\alpha_k}^{\beta_k} \cos k\omega \, d\omega,$$

for $k = 1, \ldots, n$.

3. THE DUAL PROBLEM

We can also associate a dual inequality with finite autocorrelation sequences. For $z \in \mathbf{R}^{n+1}$, we write $z \succeq_\star 0$ if

$$x^T z \geq 0 \ \ \forall x \succeq 0.$$

From the definition (2), it is clear that $z \succeq_\star 0$ if and only if $v^T Z(z) v \geq 0$ for all $v \in \mathbf{R}^{n+1}$, where

$$Z(z) = \begin{bmatrix} 2z_0 & z_1 & \cdots & z_n \\ z_1 & 2z_0 & \cdots & z_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_n & z_{n-1} & \cdots & 2z_0 \end{bmatrix}.$$

In other words, $z \succeq_\star 0$ if and only if the Toeplitz matrix $Z(z)$ is positive semidefinite.

The Lagrange dual of the cone program (7) is defined as

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{k=1}^{L} g_k^T z_k \\ \text{subject to} & \displaystyle\sum_{k=1}^{L} F_k^T z_k + c = 0 \\ & z_k \succeq_\star 0, \quad k = 1, \ldots, L. \end{array} \qquad (11)$$

The dual variables are the L vectors $z_k \in \mathbf{R}^{n_k}$. It can be shown that if the primal problem (7) is strictly feasible, then the optimal values of (7) and (11) are equal.

As an example, the dual of problem (10) is

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{k=1}^{N} (L_k^2 v_{k,0} - U_k^2 w_{k,0}) \\ \text{subject to} & \displaystyle\sum_{k=1}^{N} A_k^T (v_k - w_k) + z = c \\ & z \succeq_\star 0, \ v_k \succeq_\star 0, \ w_k \succeq_\star 0, \ \ k = 1, \ldots, L. \end{array} \qquad (12)$$

The variables are $z \in \mathbf{R}^{n+1}$ and $v_k, w_k \in \mathbf{R}^{n+1}$.

4. DUAL BARRIER FUNCTION

Several widely used interior-point methods for convex optimization rely on *barrier functions* associated with the inequality constraints. A suitable barrier for the constraint $z \succeq_\star 0$ is the standard log-det barrier

$$\phi^\star(z) = \begin{cases} -\log\det Z(z) & Z(z) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

It has been shown in [11] that this barrier, its gradient and Hessian may be computed in $O(n^3)$ floating point operations (flops), by taking advantage of the Toeplitz structure. We will find the following two properties useful in the next section. For all $y \succ_\star 0$,

$$\nabla \phi^\star(y) \prec 0, \quad y^T \nabla \phi^\star(y) = -(n+1). \qquad (13)$$

We can use this barrier function to solve the dual problem efficiently using a barrier method [12].

5. SOLVING THE PRIMAL VIA THE DUAL

If we minimize a weighted sum of the dual objective in (11) and the dual barrier functions, *i.e.*, we solve

$$\begin{array}{ll} \text{minimize} & -t \displaystyle\sum_{k=1}^{L} g_k^T z_k + \sum_{k=1}^{L} \phi^\star(z_k) \\ \text{subject to} & \displaystyle\sum_{k=1}^{L} F_k^T z_k + c = 0 \end{array} \qquad (14)$$

where $t > 0$ is a parameter, then the minimizer $(z_1, \ldots, z_L)$ satisifies the following optimality conditions:

$$-t g_k + \nabla \phi^\star(z_k) - F_k y = 0, \ \ k = 1, \ldots, L$$

for some $y \in \mathbf{R}^{n+1}$. Therefore, from (13) we have

$$F_k(y/t) + g_k = \nabla \phi^\star(z_k)/t \prec 0, \ \ k = 1, \ldots, L.$$

In other words, $x = y/t$ is strictly feasible for the primal problem (7). We can use the second equation of (13) to evaluate the duality gap between this primal feasible point $x = y/t$ and the minimizer $(z_1, \ldots, z_L)$ of (14), *i.e.*, the difference between the primal objective evaluated at x and the dual objective evaluated at $(z_1, \ldots, z_L)$:

$$\begin{aligned} c^T(y/t) - \sum_{k=1}^{L} g_k^T z_k &= -\sum_{k=1}^{L} z_k^T \left(F_k(y/t) + g_k\right) \\ &= -\sum_{k=1}^{L} z_k^T \nabla \phi^\star(z_k)/t \\ &= (n+1)L/t. \end{aligned}$$

This means that $(n+1)L/t$ is a bound on how suboptimal $x = y/t$ is for the primal problem (7). In summary, we can find primal feasible points so that the primal cost is within ϵ of optimality by minimizing (14) with $t \geq (n+1)L/\epsilon$. Problem (14) is a smooth, convex optimization problem with equality constraints, and can be efficiently solved using Newton's method. A widely used interior-point method known as SUMT is based on this idea, but solves (14) for a sequence of increasing values of t until $t \geq (n+1)L/\epsilon$, where ϵ is the desired accuracy [13]. This often requires a smaller total number of Newton iterations, than solving (14) directly for $t = (n+1)L/\epsilon$.

6. NUMERICAL EXAMPLE

Figure 1 shows the magnitude response of a length $n = 24$ bandpass filter designed by the methods introduced in this paper. The first stopband is the interval $[0, 0.2\pi]$, with an upper bound constraint of -13.2dB. The passband is the interval $[0.25\pi, 0.45\pi]$, with passband gain constrained to ± 0.5dB. The second stopband is the interval $[0.52\pi, \pi]$, with an upper bound constraint of -23dB. We minimize the weighted sum of the two stopband squared errors, with weights inversely proportional to the stopband widths.

The dual problem (12) was solved using a Matlab implementation of SUMT. A solution to the primal problem was obtained from the dual, as explained in §5. A cepstral method for spectral factorization was used to recover the filter coefficients from the autocorrelation coefficients.

During this experiment, we observed that, depending on α and β, the condition of the matrix $A(\alpha, \beta)$ deteriorates with increasing n. We plan to investigate this phenomenon further in future work.

7. CONCLUSIONS

We have formulated FIR filter magnitude constraints as generalized inequalities with respect the cone of finite autocorrelation sequences, and outlined efficient interior-point methods for solving the resulting convex optimization problems. The advantage of this approach is that sampling is avoided and so the constraints are met exactly. The method is computationaly less expensive than methods based on general-purpose SDP solvers. The method can be extended to related problems, such as magnitude design of IIR filters.

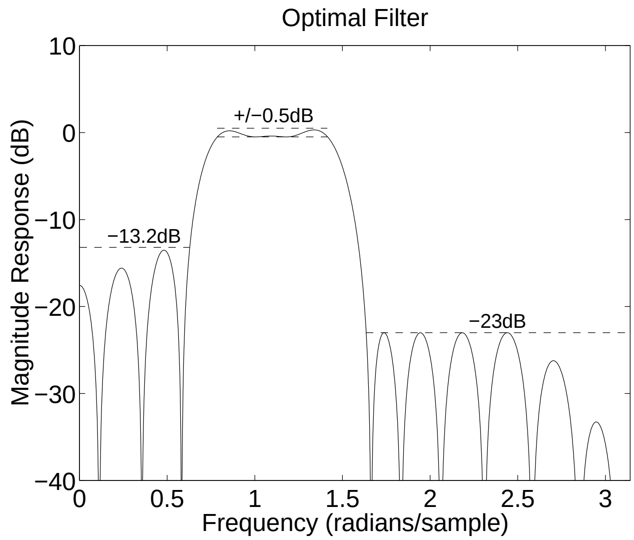

Fig. 1: Length 24 bandpass filter.

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