Semidefinite programming duality and linear system theory: connections and implications for computation

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Abstract

Several important problems in control theory can be reformulated as semidefinite programming problems (SDPs), *i.e.*, as convex optimization problems with linear matrix inequality (LMI) constraints. From duality theory in convex optimization, dual problems can be derived for these SDPs. These dual problems can in turn be reinterpreted in control or system theoretic terms, often yielding new results or new proofs for existing results from control theory. We explore such connections for a few problems associated with linear time-invariant systems. Specifically, we discuss the following three applications of SDP duality.

- Theorems of alternatives provide systematic and unified proofs of *necessary and sufficient conditions for solvability* of LMIs. As an example, we present a simple new proof of the KYP lemma.
- The dual problem associated with an SDP can be used to derive *lower bounds* on the optimal value. As an example, a duality-based proof of the Enns-Glover lower bound.
- The optimal solution of an SDP is characterized by *necessary and sufficient optimality conditions* that involve the dual variables. As an example, we show that the properties of the solution of the LQR problem can be derived directly from the SDP optimality conditions.

Several of the results that we use from convex duality require technical conditions (so-called *constraint qualifications*). We show that for problems involving Riccati inequalities these constraint qualifications are related to controllability and observability. In particular, this will lead us to a new criterion for controllability. We also point out some implications of these results for computational methods for large-scale SDPs arising in control.

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1 Introduction

Over the past few years, convex optimization, and semidefinite programming (SDP) in particular, have come to be recognized as a valuable tool for control system analysis and design via numerical methods. Convex optimization problems enjoy a number of advantages over more general optimization problems:

- Every stationary point is also a global minimizer.
- There are very efficient algorithms for computing the global minimizer, *i.e.*, algorithms that rapidly compute the global optimum, with non-heuristic stopping criteria. Convex optimization problems are also very tractable from a theoretical standpoint, since their worst-case complexity is polynomial.
- There is a well-developed duality theory.

Thus far, the main motivation behind reformulating systems and control problems into SDP problems has been in obtaining numerical solutions as well as to develop computer-aided analysis and design tools [11, 12, 24]. In this paper, we explore another application of semidefinite programming in systems and control: We discuss the application of semidefinite programming duality theory in systems and control to obtain new theoretical insight or to provide new proofs to existing results, as well as towards developing more efficient algorithms for the solution of SDP problems underlying control.

2 Linear matrix inequalities

2.1 Definition

A linear matrix inequality is a constraint of the form

$$F_0 + x_1 F_1 + \cdots + x_m F_m \geq 0 \qquad (1)$$

where the variable is $x \in \mathbf{R}^m$ and the problem data are the matrices $F_i = F_i^T \in \mathbf{R}^{n \times n}$. The inequality sign denotes matrix inequality, *i.e.*, $A \geq 0$ means A is positive semidefinite. We will also consider strict matrix inequalities

$$F_0 + x_1 F_1 + \cdots + x_m F_m > 0, \qquad (2)$$

where the inequality means the lefthand side is a positive definite matrix.

2.2 Theorems of alternatives

Theorems of alternatives give necessary and sufficient conditions for solvability of an LMI. We consider two variations [5, 8, 14, 23].

Theorem 1 *The strict LMI (2) is feasible if and only there does not exist a* $Z = Z^T \in \mathbf{R}^{n\times n}$ *that satisfies*

$$0 \neq Z \geq 0, \quad \mathbf{Tr}\, F_0 Z \leq 0, \quad \mathbf{Tr}\, F_i Z = 0,\ i = 1, \ldots, m. \qquad (3)$$

The conditions (2) and (3) are called alternatives. The theorem states that exactly one of both alternatives is feasible.

A similar theorem holds for nonstrict LMIs, but it requires a technical condition.

Theorem 2 *Assume*

$$\sum_{i=1}^{m} y_i F_i \geq 0 \Longrightarrow \sum_{i=1}^{m} y_i F_i = 0. \qquad (4)$$

Then the LMI (1) is feasible if and only there does not exist a $Z = Z^T \in \mathbf{R}^{n\times n}$ *that satisfies*

$$Z \geq 0, \quad \mathbf{Tr}\, F_0 Z < 0, \quad \mathbf{Tr}\, F_i Z = 0,\ i = 1, \ldots, m.$$

The condition (4) is called a *constraint qualification*. The need for this condition can be explained geometrically as follows. The theorem is an application of a fundamental result in convex analysis, which states that if a point is not an element of a closed convex set, then it can be separated from it by a hyperplane. We apply this result to the convex set $\mathcal{A}$ defined as

$$\mathcal{A} = \left\{ U = U^T \;\middle|\; \exists x : F_0 + \sum_i x_i F_i \geq U \right\}.$$

The LMI (1) is infeasible if and only if $0 \notin \mathcal{A}$, and the dual variable Z can be interpreted as the normal to a hyperplane separating $\mathcal{A}$ and the origin. The constraint qualification implies that $\mathcal{A}$ is closed, and we can apply the separating hyperplane theorem.

2.3 Riccati inequalities

It is well-known that for a linear time-invariant system, checking whether or not a number of important properties (*e.g.*, stability and passivity) hold is equivalent to the feasibility of Lyapunov or Riccati equations. The two theorems of alternatives we mentioned (or one of their many variations) can be used to derive necessary and sufficient conditions for the infeasibility of Lyapunov or Riccati equations. These conditions, when reinterpreted in the context of the original LTI system, yield new proofs for existing results in control theory. We will demonstrate this with a new proof of the well-known Kalman-Yakubovich-Popov (KYP) lemma, which relates feasibility of an LMI to a frequency-domain condition. Note that our statement of the KYP lemma in the following theorem is slightly different (and more general) than existing versions (see [18] and the references therein).

Theorem 3 *The LMI*

$$\begin{bmatrix} A^T P + PA - M & PB \\ B^T P & -I \end{bmatrix} < 0 \qquad (5)$$

in the variable $P = P^T$*, is feasible if and only if for all* ω*,*

$$(j\omega I - A)u = Bv,\ (u, v) \neq 0 \Longrightarrow v^* v + u^* M u > 0. \qquad (6)$$

We first show that feasibility (5) implies that (6) holds for all ω. Suppose that P satisfies (5), and that (u, v) satisfy $(j\omega I - A)u = Bv$. It is easily verified that

$$-\begin{bmatrix} u^* & v^* \end{bmatrix} \begin{bmatrix} A^T P + PA - M & PB \\ B^T P & -I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = v^* v + u^* M u.$$

Therefore $v^* v + u^* M u > 0$ if $(u, v) \neq 0$.

Next, we prove that if (5) is infeasible, then (6) does not hold for some ω. Applying theorem 1, we know that if (5) is infeasible, then there exists a $Z = Z^T \neq 0$, such that

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \geq 0, \qquad (7)$$

$$Z_{11} A^T + A Z_{11} + Z_{12} B^T + B Z_{12}^T = 0, \qquad (8)$$

$$\mathbf{Tr}\, M Z_{11} + \mathbf{Tr}\, Z_{22} \leq 0. \qquad (9)$$

We will describe how to construct from Z a frequency ω at which (6) does not hold. We can assume without loss of generality that

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^T & V^T \end{bmatrix}$$

where U has full rank. In terms of U and V we can express (8)–(9) more simply as follows:

$$\mathbf{Tr}\, U^T M U + \mathbf{Tr}\, V^T V \leq 0, \qquad (10)$$

and $AUU^T + BVU^T$ is skew-symmetric, *i.e.*, it can be written as

$$AUU^T + BVU^T = USU^T$$

where S is skew-symmetric. Since U has full rank, this last equation implies

$$AU + BV = US. \tag{11}$$

The next step is to take the Schur decomposition of S: $S = \sum_i j\omega_i q_i q_i^*$ where $\sum_i q_i q_i^* = I$. From (10) we obtain

$$\sum_i q_i^* \left(U^T MU + V^T V\right) q_i \leq 0.$$

At least one of the terms in this expression must be less than or equal to zero. Let k be the index of that term, and define $\tilde{u} = Uq_k$, $\tilde{v} = Vq_k$. ($\tilde{u}$ is nonzero because U has full rank.) We have

$$\tilde{u}^* M\tilde{u} + \tilde{v}^*\tilde{v} \leq 0$$

and, by multiplying (11) with q_k on the right,

$$A\tilde{u} + B\tilde{v} = j\omega_k\tilde{u}.$$

In other words we have constructed a $\tilde{u}$ and $\tilde{v}$ showing that (6) does not hold at $\omega = \omega_k$.

Remarks:

- We make no assumptions on M, A, and B. In particular, the matrix M can be indefinite, and (A, B) can be uncontrollable, or have uncontrollable modes on the imaginary axis.

  Two important special cases are $M = -C^T C$, which gives the H_∞ Riccati inequality, and $M > 0$, which yields the LQR Riccati inequality.

- If $(j\omega I - A)$ is invertible, then (6) is equivalent to

  $$I + B^T(-j\omega I - A^T)^{-1} M(j\omega I - A)^{-1} B > 0, \tag{12}$$

  so both conditions are equivalent if A has no imaginary eigenvalues. However, if A has imaginary eigenvalues, then the conditions are different, *i.e.*, requiring that (6) holds for all ω, is not the same as requiring that (12) holds for all ω where $j\omega I - A$ is invertible.

  As an example, consider

  $$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = 0, \quad M = -I.$$

  The LMI (5) is infeasible, and the frequency condition (6) does not hold at $\omega = 1$ (choose $u = (1, j)$, $v = 0$). However the inequality (12) is clearly valid for all ω where $j\omega I - A$ is invertible.

We can also derive a similar characterization for the non-strict Riccati inequality.

Theorem 4 *Suppose (A, B) is controllable. Then the LMI*

$$\begin{bmatrix} A^T P + PA - M & PB \\ B^T P & -I \end{bmatrix} \leq 0 \tag{13}$$

in the variable $P = P^T$, is feasible if and only if for all ω,

$$(j\omega I - A)u = Bv \Longrightarrow v^* v + u^* Mu \geq 0. \tag{14}$$

The proof is similar to the proof of theorem 3, but instead of theorem 3 we apply theorem 2. All we have to do is show that controllability implies that the constraint qualification in theorem 2 holds. This follows from the following result.

Lemma 1 *(A, B) is controllable if and only if*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \leq 0 \tag{15}$$

implies $P = 0$.

We can prove this as follows. First, assume (A, B) is controllable and P satisfies (15), *i.e.*,

$$A^T P + PA \leq 0, \quad PB = 0.$$

Since (A, B) is controllable, there exists a K such that $A + BK$ is stable. Therefore $(A + BK)^T P + P(A + BK) = Q$ where $Q = A^T P + PA \leq 0$. Since $A + BK$ is stable, we can express P as

$$P = -\int_0^\infty e^{(A+BK)^T t} Q e^{(A+BK)t} dt.$$

This expression, together with $B^T PB = 0$, implies $Q = 0$, and hence $P = 0$. This shows that the only solution to (15) is $P = 0$.

Conversely, suppose (A, B) is not controllable, *i.e.*, there exists a nonzero v with $v^* A = \lambda v^*$, $v^* B = 0$. Then $P = \pm\mathbf{Re}(vv^*)$ satisfies

$$A^T P + PA = \pm 2\mathbf{Re}(\lambda)P, \quad PB = 0,$$

i.e., (15) has a nonzero solution.

3 The dual SDP

3.1 Definition

The problem of minimizing a linear function subject to an LMI constraint, *i.e.*,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + \cdots + x_m F_m \geq 0, \end{array} \tag{16}$$

is called a *semidefinite programming* problem (SDP). From convex duality, we can associate with an SDP a dual problem

$$\begin{array}{ll} \text{maximize} & -\mathbf{Tr}\, F_0 Z \\ \text{subject to} & \mathbf{Tr}\, F_i Z = c_i, \quad i = 1, \ldots, m \\ & Z = Z^T \geq 0 \end{array} \tag{17}$$

where the variable is the matrix $Z = Z^T$.

Let $p^\star$ and $d^\star$ be the optimal values of (16) and (17), respectively. (We allow values $\pm\infty$: $p^\star = +\infty$ if the primal problem is infeasible and $p^\star = -\infty$ if it is unbounded below, with a similar convention for $d^\star$.) From convex duality we have the following relation between $p^\star$ and $d^\star$.

- $p^\star \geq d^\star$. This is called *weak duality.*
- *Strong duality, i.e.*, $p^\star = d^\star$, holds if the primal problem is strictly feasible (*i.e.*, there exists an x with $F(x) > 0$), or the dual problem is strictly feasible (*i.e.*, there exists a $Z > 0$ with $\mathbf{Tr}\, F_i Z = c_i$, $i = 1, \ldots, m$).

As a result, we can use the dual problem to derive and prove lower bounds on the optimal value $p^\star$.

3.2 Lower bounds on the H_∞-norm

As an example, consider the LMI

$$\begin{array}{ll} \text{minimize} & \beta \\ \text{subject to} & \begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\beta I \end{bmatrix} \leq 0 \end{array}$$

with variables β, $P = P^T$. It can be shown that the optimal value is $p^\star = \|H\|_\infty^2$ where $H(s) = C(sI - A)^{-1}B$ (see [7, p. 91]).

The dual problem is

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}\, C Z_{11} C^T \\ \text{subject to} & Z_{11} A^T + A Z_{11} + Z_{12} B^T + B Z_{12}^T = 0 \\ & Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \geq 0, \quad \mathbf{Tr}\, Z_{22} = 1 \end{array} \tag{18}$$

with variable $Z = Z^T$. Any dual feasible Z provides a lower bound $\sqrt{\mathbf{Tr}\, C Z_{11} C^T}$ for $\|H\|_\infty$ [4]. As an illustration, we can verify the Enns-Glover lower bound

$$\|H\|_\infty \geq \sigma_{\max} \tag{19}$$

where $\sigma_{\max}$ is the maximum Hankel singular value (see [13, 9]). We can prove this result by constructing a dual feasible Z as follows.

- Calculate the controllability and observability Gramians W_c, W_o from
$$A W_c + W_c A^T + B B^T = 0,$$
$$A^T W_o + W_o A + C^T C = 0.$$
- Let $\sigma_{\max} = (\lambda_{\max}(W_c W_o))^{1/2}$ be the largest Hankel singular value, and let z be the corresponding eigenvector of $W_c^{1/2} W_o W_c^{1/2}$.
- Compute X, Y from
$$AY + YA^T + W_c^{1/2} z z^T W_c^{1/2} = 0,$$
$$A^T X + XA + W_c^{-1/2} z z^T W_c^{-1/2} = 0.$$
- Let $Z_{11} = Y + W_c X W_c$, $Z_{12} = W_c X B$, $Z_{22} = B^T X B$.

It can be verified that Z is feasible in (18) with

$$\mathbf{Tr}\, C Z_{11} C^T \geq \mathbf{Tr}\, C Y C^T = \sigma_{\max}^2,$$

which proves the bound (19).

4 Optimality conditions

4.1 Complementary slackness

The following facts are useful when studying the properties of the optimal solutions of the primal and dual SDP.

- If the primal problem is strictly feasible, then $p^\star = d^\star$ and the dual optimum is attained, *i.e.*, there exists a dual optimal Z.
- If the dual problem is strictly feasible, then $p^\star = d^\star$ and the primal optimum is attained, *i.e.*, there exists a primal optimal x.
- If strong duality holds, then a primal feasible x and a dual feasible Z are optimal if and only if $F(x)Z = 0$. This property is called *complementary slackness.*

These properties allow us to state necessary and sufficient conditions for optimality. For example, it follows that if the primal problem is strictly feasible, then a primal feasible x is optimal if and only if there exists a dual feasible Z with $F(x)Z = 0$.

4.2 The LQR problem

We consider the following stochastic formulation of the LQR problem. For the system

$$\dot{x} = Ax + Bu,$$

with the initial condition $x(0)$ being a random vector satisfying $\mathbf{E}\left(x(0)x(0)^T\right) = I$, we want to find the input u that minimizes

$$J = \mathbf{E}\left(\int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt\right).$$

We assume $Q \geq 0$, $R > 0$, (Q, A) is observable, and (A, B) controllable[1]. Let $J_{\rm opt}$ denote the minimum value of the LQR problem.

We can associate with the LQR problem the following SDP:

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}\, P \\ \text{subject to} & \left[\begin{array}{cc} A^TP+PA+Q & PB \\ B^TP & R \end{array}\right] \geq 0. \end{array} \tag{20}$$

The variable is $P = P^T$. The interpretation is as follows. If P is feasible in (20) then for all x that satisfy

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and $x(t) \to 0$ for $t \to \infty$, we have

$$x_0^T P x_0 \leq \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt$$

or

$$\mathbf{Tr}\, P \leq \mathbf{E}\left(\int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt\right).$$

Therefore, the objective value of (20) is a lower bound on the optimal LQR cost $J_{\rm opt}$. The optimal value of the SDP is the best lower bound that can be obtained.

The dual of the SDP (20) is

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, QZ_{11} + \mathbf{Tr}\, RZ_{22} \\ \text{subject to} & AZ_{11} + Z_{11}A^T + BZ_{12}^T + Z_{12}B^T + I = 0 \\ & Z = Z^T = \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{array}\right] \geq 0. \end{array} \tag{21}$$

The dual variable Z can be interpreted as follows. Without loss of generality we can assume that Z can be expressed as

$$Z = \left[\begin{array}{c} I \\ K \end{array}\right] \tilde{Z} \left[\begin{array}{cc} I & K^T \end{array}\right]$$

so we can reformulate the problem as a problem in $\tilde{Z}$ and K:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\,(Q + K^TRK)\tilde{Z} \\ \text{subject to} & (A+BK)\tilde{Z} + \tilde{Z}(A+BK)^T + I = 0 \\ & \tilde{Z} \geq 0. \end{array} \tag{22}$$

[1] This is a variation on the problem posed in [6, §12.1]), where the underlying system is

$$\dot{x} = Ax + Bu + w,$$

w is a white noise process with unit power spectral density, and the goal is to find u that minimizes $J = \lim_{t\to\infty} \mathbf{E}\left(x(t)^TQx(t) + u(t)^TRu(t)\right)$. It can be shown that this problem is equivalent to the problem considered in this paper.

The objective function is equal to the LQR cost

$$\mathbf{E}\left(\int_0^\infty x(t)^T(Q + K^TRK)x(t)dt\right)$$

that results when we apply a linear state feedback $\dot{x} = (A + BK)x$. Clearly, for every K this yields an upper bound on the optimal LQR cost; by minimizing over $\tilde{Z}$ and K we obtain the upper bound, achievable using state-feedback.

Let us now apply the optimality conditions to the pair of SDPs (20) and (21). It can be shown that if (Q, A) is observable, then the primal problem is strictly feasible, and if (A, B) is controllable, then the dual problem is strictly feasible. We therefore have strong duality $p^\star = d^\star$, and the primal and dual optima are attained.

The complementary slackness conditions that relate optimal P and Z, are given by

$$\left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{array}\right] \left[\begin{array}{cc} A^TP+PA+Q & PB \\ B^TP & R \end{array}\right] = 0,$$

i.e.,

$$\left[\begin{array}{c} I \\ K \end{array}\right] \tilde{Z} \left[\begin{array}{cc} I & K^T \end{array}\right] \left[\begin{array}{cc} A^TP+PA+Q & PB \\ B^TP & R \end{array}\right] = 0,$$

From the first constraint in (22) it is clear that $\tilde{Z} > 0$. If $\tilde{Z}$ were singular, we would have a contradiction, since

$$v^T((A+BK)\tilde{Z} + \tilde{Z}(A+BK)^T + I)v = v^Tv$$

for all v with $\tilde{Z}v = 0$. Therefore,

$$\left[\begin{array}{cc} I & K^T \end{array}\right] \left[\begin{array}{cc} A^TP+PA+Q & PB \\ B^TP & R \end{array}\right] = 0,$$

from which we immediately obtain two well-known properties of the solution of the LQR problem: the optimal input is a constant state feedback given by

$$K = -R^{-1}B^TP,$$

and, secondly, P satisfies the algebraic Riccati equation

$$A^TP + PA + Q - PBR^{-1}B^TP = 0.$$

5 Conclusions and implications for computation

We have presented simple proofs of several well-known facts related to Riccati inequalities. The proofs rely on a few basic theorems from SDP duality. While we believe these proofs (and in some cases the results) are new, this work is mainly motivated by the hope that computational methods for LMI problems in control can benefit from insight in the dual problems, for the following reasons.

– Modern interior-point methods for semidefinite programming are primal-dual methods. Some of these methods require primal and dual feasible starting points. A system-theoretical interpretation of the dual problem is helpful when selecting good dual feasible starting points.

  For infeasible methods, *i.e.*, methods that do not require primal and dual feasible starting points, insight in the dual problem is important when the algorithm terminates with the conclusion that the problem is not dual feasible, and we want to understand why.

– Decomposition methods in large-scale linear programming, *i.e.*, methods that break up a problem into smaller independent problems and then combine the results, rely heavily on duality. It would be very interesting if similar decomposition schemes could be formulated for large SDP problems in control.

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