A polynomial-time algorithm for determining quadratic Lyapunov functions for nonlinear systems

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We consider nonlinear systems $dx/dt = f(x(t))$ where $Df(x(t))$ is known to lie in the convex hull of L matrices $A_1, \ldots, A_L \in \mathbf{R}^{n\times n}$. For such systems, quadratic Lyapunov functions can be determined using convex programming techniques [1]. This paper describes an algorithm that either finds a quadratic Lyapunov function or terminates with a proof that no quadratic Lyapunov function exists. The algorithm is an interior-point method based on the theory developed by Nesterov and Nemirovsky [2].

1. AN EQUIVALENT OPTIMIZATION PROBLEM

Quadratic Lyapunov functions of the form x^TPx can be determined by solving a set of matrix inequalities for P:

$$\begin{array}{l} A_k^TP + PA_k \leq 0, \quad k = 1, \ldots, L \\ \kappa I \leq P \leq I, \end{array} \tag{1}$$

where I is the $n \times n$ identity matrix, and the notation $X \geq 0$ means that X is positive semidefinite. The lower and upper bounds on P are added for numerical reasons. These bounds limit the condition number of P to $1/\kappa$, which is not very restrictive as long as κ is a small number. The upper bound also guarantees that the set of feasible matrices P is bounded.

Problem (1) can be converted into an optimization problem by adding an artificial variable t:

$$\begin{array}{ll} \text{minimize} & t \\ \text{such that} & -A_k^TP - PA_k + tF_k \geq 0, \quad k = 1, \ldots, L \\ & -P + tF_{L+1} + I \geq 0 \\ & P + tF_{L+2} - \kappa I \geq 0 \\ & t \geq 0 \end{array} \tag{2}$$

The optimization algorithm requires a feasible solution to start with. Assume one has a symmetric matrix P_0 as initial guess for the matrix P. One can then choose $F_k =$

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$I + A_k^T P_0 + P_0 A_k$, $k = 1, \ldots, L$, $F_{L+1} = P_0$ and $F_{L+2} = (1+\kappa)I - P_0$. This implies that $t = 1$, $P = P_0$ satisfies the constraints (2). Starting with these initial values, one minimizes t. If the minimum value of t is zero, a solution to the original feasibility problem (1) has been found. If on the other hand the minimum value of t is greater than zero, we have actually proven that the original problem is infeasible.

2. POTENTIAL FUNCTION

A number of very efficient methods have been recently developed for solving optimization problems over the cone of positive definite matrices. These so-called *interior-point methods* were first introduced for linear programming by Karmarkar [3]. Since then, they have been extended to general convex problems. The first unified treatment can be found in [2]. It is now possible to solve problems involving large matrix inequalities in a reasonable amount of time. The worst-case effort can be shown to grow polynomially with problem size; the performance is even better in practice.

The most efficient interior-point methods are based on a potential function. For problem (2) one can use

$$\psi(P,t) = q \log t - \sum_{k=1}^{L+2} \log\det X_k \tag{3}$$

where

$$\begin{aligned} X_k &\triangleq -A_k^T P - P A_k + tF_k, \quad k = 1, \ldots L \\ X_{L+1} &\triangleq -P + tF_{L+1} + I, \\ X_{L+2} &\triangleq P + tF_{L+2} - \kappa I, \end{aligned}$$

and q is a positive scalar. The potential function $\psi(P,t)$ consists of two parts. The first term, $q \log t$, is concave and rewards a decrease in t. The second term, $-\sum_{k=1}^{L+2} \log\det X_k$, acts as a barrier for the feasible set.

Although the function ψ itself is not convex, it can be shown that $\exp\psi$ is convex if $q > n(L+2)$. This property has the important consequence that local minima of ψ will necessarily be unique and global.

3. MINIMIZATION OF THE POTENTIAL FUNCTION

Interior-point methods use a damped Newton method for minimizing the potential function (see [2]). Skipping details, this amounts to repeatedly computing a step δP, δt by solving the overdetermined system:

$$\begin{aligned} I &\approx X_k^{-1/2} \left(-A_k^T \delta P - \delta P A_k + \delta t F_k \right) X_k^{-1/2}, \quad k = 1, \ldots, L \\ I &\approx X_{L+1}^{-1/2} \left(-\delta P + \delta t F_{L+1} \right) X_{L+1}^{-1/2} \\ I &\approx X_{L+2}^{-1/2} \left(\delta P + \delta t F_{L+2} \right) X_{L+2}^{-1/2} \\ -q &\approx t^{-1} \delta t \end{aligned} \tag{4}$$

in a least-squares sense. One then takes a suitably scaled step in this direction.

If the problem is feasible, it can be shown that this Newton step reduces the potential function at least by an absolute constant in each iteration. This property is crucial in establishing a proof of polynomiality of the algorithm [2].

4. DETECTION OF INFEASIBILITY

If the potential function is not unbounded below, but reaches its minimum for some (P^*, t^*) with $t^* > 0$, we may conclude that the problem (1) is infeasible. A formal proof of infeasibility can be derived from duality theory [2]. It consists in generating a set of $n \times n$ matrices $Z_k \geq 0$, $k = 1, \ldots, L+2$ that satisfy:

$$\begin{aligned}
&\mathbf{Tr}\ Z_{L+1} - \kappa \mathbf{Tr}\ Z_{L+2} < 0 \\
&-\sum_{k=1}^{L} \left(Z_k A_k^T + A_k Z_k \right) - Z_{L+1} + Z_{L+2} = 0 \\
&\sum_{k=1}^{L+2} \mathbf{Tr}\ F_k Z_k = 1.
\end{aligned}$$

A set of matrices Z_k satisfying these conditions can be derived from the minimizers (P^*, t^*) at a negligible computational cost.

5. NUMERICAL EXAMPLE

Interior-point methods typically require a small number of iterations, but each iteration involves solving a large least-squares problem. There is no need, however, to solve these least-squares problems exactly. Approximate solutions can already yield descent directions that are sufficient for polynomiality [4]. The use of iterative methods such as conjugate gradients and LSQR [5] to approximately solve the least-squares problems can therefore lead to very considerable savings in computer time.

The numerical data for Fig. 1 were obtained as follows. We randomly generate 20×20 matrices δA_k, $k = 1, \ldots, 50$, and construct problems of the form (1) by taking $A_k = -I + \alpha\ \delta A_k, k = 1, \ldots, L$.

The problem will clearly be feasible for small values of α and infeasible for larger values. Figure 1 shows the total cpu-time (on a SUN 4/670) and the number of iterations for different values of α. From these data, we note that the algorithm is fast if the problem is clearly feasible or infeasible, and becomes slower as we approach the boundary between feasibility and infeasibility.

In this example, the optimization problem (2) has 211 unknowns (the scalar t plus the elements of the symmetric matrix P). The least-squares problem (4) that has to be solved in each iteration has size 10921×211.

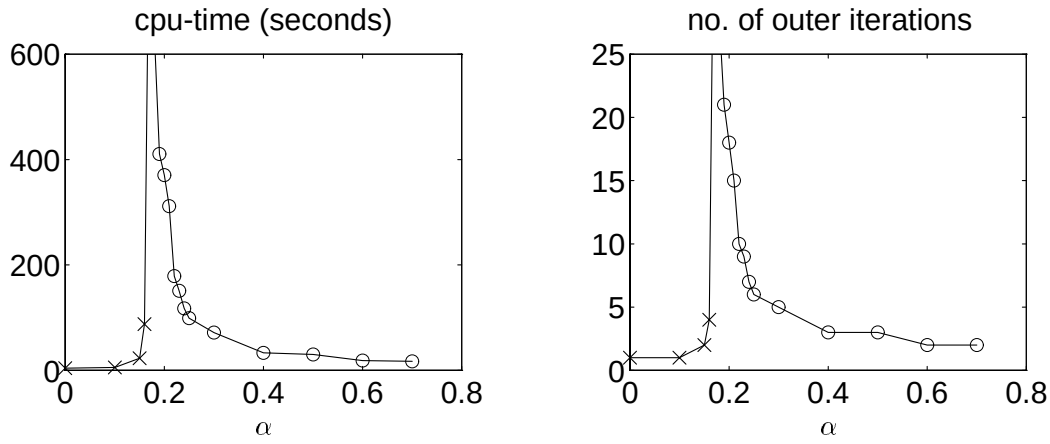

Figure 1. Total cpu-time and number of iterations for the experiment described in Section 5. The instances marked with '×' were feasible; the instances marked with 'o' were infeasible.

REFERENCES

1. S. Boyd and Q. Yang. Structured and simultaneous Lyapunov functions for system stability problems. *Int. J. Control*, 49(6):2215-2240, 1989.
2. Yu. Nesterov and A. Nemirovsky. *Interior point polynomial methods in convex programming: Theory and applications.* SIAM, 1993.
3. N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373-395, 1984.
4. L. Vandenberghe and S. Boyd. Primal-dual potential reduction method for problems involving matrix inequalities. To be published in *Mathematical Programming*, 1993.
5. C.C. Paige and M.A. Saunders. LSQR: An algorithm for sparse linear equations and sparse linear least squares. *ACM Transactions on Mathematical Software*, 8(1):43-71, 1982.