Generalized KYP Lemma with Real Data

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***Abstract*—A recent generalization of the Kalman-Yakubovich-Popov (KYP) lemma establishes the equivalence between a semi-infinite inequality on a segment of a line or circle in the complex plane and a linear matrix inequality (LMI). In this paper we show that when the data are real, the matrix variables in the LMI can be restricted to be real, even when the frequency range is asymmetric with respect to the real axis.**

I. Introduction

The Kalman-Yakubovich-Popov (KYP) lemma [1]–[3] is a key result in modern system and control theory. The classical version of the lemma, which states the equivalence between a semi-infinite frequency domain inequality (FDI) on the entire frequency axis and a finite-dimensional linear matrix inequality (LMI), was recently extended to FDIs on frequency intervals by Iwasaki and Hara [4], and Scherer [5]. One version of their generalized KYP (gKYP) lemma states that, given matrices A, B, and a Hermitian matrix Θ, the FDI

$$\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix}^H \Theta \begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \prec 0 \qquad (1)$$

holds for all $\omega \in [\omega_1, \omega_2]$ if and only if there exist Hermitian matrices P, Q such that

$$Q \succ 0\,, \qquad (2a)$$

$$\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^H \left(\begin{bmatrix}0 & P\\ P & 0\end{bmatrix} - \begin{bmatrix}Q & -j\omega_{\mathrm{m}}Q\\ j\omega_{\mathrm{m}}Q & \omega_1\omega_2 Q\end{bmatrix}\right)\begin{bmatrix}A & B\\ I & 0\end{bmatrix} + \Theta \prec 0\,, \qquad (2b)$$

where $\omega_{\mathrm{m}} = (\omega_1 + \omega_2)/2$. This result extends the classical KYP lemma, which states that the FDI (1) holds on the entire frequency axis $\mathbf{R} \cup \{\infty\}$ if and only if there exists a P that satisfies (2b) with $Q = 0$. We will refer to (2) as a gKYP-LMI.

In many applications of the classical or generalized KYP lemmas the matrices A, B, and Θ are real (see e.g. [6]–[10]). In this case, the matrix variable P in the classical KYP lemma may be taken to be real symmetric without loss of generality. In the gKYP-LMI (2), on the other hand, the coefficient of Q is complex if the frequency interval $[\omega_1, \omega_2]$ is asymmetric around the real axis, i.e., $\omega_{\mathrm{m}} \neq 0$. Therefore, as pointed out in [4, p.48–49] and [11, p.313], the LMI seems to require complex Hermitian matrix variables P and Q. The purpose of this paper is to show that, contrary to this intuition, the matrix variables P, Q in the gKYP-LMI may be constrained to be real when the data A, B, and Θ are real, even if the frequency domain is asymmetric around the real axis. This has computational advantages because it reduces the number of variables in the gKYP-LMI.

The paper is organized as follows. Section II provides the required background on the gKYP lemma. Section III presents preliminary material needed in the proof of the main result in Section IV. Section V concludes the paper.

Notation: The set of real symmetric $n \times n$ matrices is denoted by $\mathbf{S}_n$ and the set of complex Hermitian $n \times n$ matrices is denoted by $\mathbf{H}_n$. For a matrix $X \in \mathbf{H}_n$ or $X \in \mathbf{S}_n$, the inequality $X \prec 0$ ($X \preceq 0$) means X is negative definite (semidefinite). The transpose of a matrix X is written as X^T, the complex conjugate as $\bar{X}$, and the complex conjugate transpose as $X^H = \bar{X}^T$. The real part of X is denoted by $\Re(X)$. The symbol $\otimes$ indicates the Kronecker matrix product. For a vector $x \in \mathbf{C}_n$, $\mathbf{diag}(x)$ denotes the diagonal matrix with diagonal entries $x_1, \ldots, x_n$, while for matrices $X, Y, \ldots$, $\mathbf{diag}(X, Y, \ldots)$ denotes the block diagonal matrix with diagonal blocks $X, Y, \ldots$

II. Generalized KYP Lemma

The generalized KYP lemma [4], [5] is concerned with inequalities on curves in the complex plane defined as

$$\mathbf{\Lambda}(\Phi, \Psi) = \left\{ \lambda \in \mathbf{C} \;\middle|\; \begin{bmatrix}\lambda\\ 1\end{bmatrix}^H \Phi \begin{bmatrix}\lambda\\ 1\end{bmatrix} = 0\,,\ \begin{bmatrix}\lambda\\ 1\end{bmatrix}^H \Psi \begin{bmatrix}\lambda\\ 1\end{bmatrix} \geq 0 \right\}, \qquad (3)$$

with Φ, $\Psi \in \mathbf{H}_2$. The set $\mathbf{\Lambda}(\Phi, \Psi)$ represents a curve in the complex plane if and only if: (i) $\det(\Phi) < 0$, so that $\mathbf{\Lambda}(\Phi, 0)$ corresponds to a circle or a straight line, and (ii) Φ and Ψ satisfy an additional condition that excludes empty or singleton sets. The latter condition is most easily expressed with the help of a congruence transformation introduced in [4]: It can be shown that if $\det(\Phi) < 0$ then there exists a nonsingular $T \in \mathbf{C}_{2\times 2}$ such that

$$\Phi = T^H \Phi_{\mathrm{o}} T\,, \qquad \Psi = T^H \Psi_{\mathrm{o}} T\,, \qquad (4a)$$

where

$$\Phi_{\mathrm{o}} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \qquad \Psi_{\mathrm{o}} = \begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}, \qquad (4b)$$

and $\alpha, \beta, \gamma \in \mathbf{R}$ with $\alpha \leq \gamma$ (see [4, Appendix 1] for a constructive proof). The set $\mathbf{\Lambda}(\Phi, \Psi)$ with $\det(\Phi) < 0$ is not empty and not a singleton if and only if either $0 \leq \alpha \leq \gamma$ or $\alpha < 0 < \gamma$. In the former case, $\mathbf{\Lambda}(\Phi, \Psi)$ coincides with the circle or straight line given by $\mathbf{\Lambda}(\Phi, 0)$, while in the latter case it is a segment of this circle or line. The imaginary axis and the unit circle are examples of (3) with $\Psi = 0$ and Φ equal to

$$\Phi_{\mathrm{c}} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \qquad \Phi_{\mathrm{d}} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \qquad (5)$$

for the imaginary axis, respectively, the unit circle.

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If $\mathbf{\Lambda}(\Phi,\Psi)$ is unbounded (i.e., $\Phi_{11}=0$ and $\Psi_{11}\geq 0$), we will extend it with ∞. In the extended definition, $\mathbf{\Lambda}(\Phi,\Psi)$ can be interpreted as consisting of elements (μ,ν) of the set

$$\mathbf{\Sigma}(\Phi,\Psi)=\Bigg\{(\mu,\nu)\in\mathbf{C}\times\mathbf{C}\;\Bigg|\;(\mu,\nu)\neq 0\,,\\ \begin{bmatrix}\mu\\ \nu\end{bmatrix}^H\Phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}=0\,,\;\begin{bmatrix}\mu\\ \nu\end{bmatrix}^H\Psi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\geq 0\Bigg\}\,.$$

If $\nu\neq 0$, then $\lambda=\mu/\nu$ is a finite point in $\mathbf{\Lambda}(\Phi,\Psi)$. If $\nu=0$, then $\mathbf{\Lambda}(\Phi,\Psi)$ includes $\lambda=\infty$. Note that the congruence transformation (4a) corresponds to a linear transformation between $(\mu,\nu)\in\mathbf{\Sigma}(\Phi,\Psi)$ and $(s,t)\in\mathbf{\Sigma}(\Phi_\mathrm{o},\Psi_\mathrm{o})$:

$$\begin{bmatrix}s\\ t\end{bmatrix}=T\begin{bmatrix}\mu\\ \nu\end{bmatrix}\,.$$

To formulate the generalized KYP lemma of [4], [5] we define

$$N_\lambda(A,B)=\{(u,v)\in\mathbf{C}_n\times\mathbf{C}_m\mid(\lambda I-A)u=Bv\}\,,$$

if $\lambda\neq\infty$, while for $\lambda=\infty$

$$N_\lambda(A,B)=\{0\}\times\mathbf{C}_m\,.$$

Theorem 1 (Generalized KYP Lemma)**.** *Let $A\in\mathbf{C}_{n\times n}$, $B\in\mathbf{C}_{n\times m}$, and $\Theta\in\mathbf{H}_{n+m}$. Suppose Φ, $\Psi\in\mathbf{H}_2$ define a curve $\mathbf{\Lambda}(\Phi,\Psi)$ in the complex plane. Then the following two statements are equivalent.*

1) *If $\lambda\in\mathbf{\Lambda}(\Phi,\Psi)$, then*
$$\begin{bmatrix}u\\ v\end{bmatrix}^H\Theta\begin{bmatrix}u\\ v\end{bmatrix}<0 \tag{6}$$
for all nonzero $(u,v)\in N_\lambda(A,B)$.
2) *There exist $P,Q\in\mathbf{H}_n$ that satisfy*
$$Q\succ 0\,,\quad\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^H(\Phi\otimes P+\Psi\otimes Q)\begin{bmatrix}A & B\\ I & 0\end{bmatrix}+\Theta\prec 0\,. \tag{7}$$

The reader is referred to [4], [5] for a proof. If A has no eigenvalues in $\mathbf{\Lambda}(\Phi,\Psi)$, then the first statement reduces to the FDI

$$\begin{bmatrix}(\lambda I-A)^{-1}B\\ I\end{bmatrix}^H\Theta\begin{bmatrix}(\lambda I-A)^{-1}B\\ I\end{bmatrix}\prec 0\,,\quad\forall\lambda\in\mathbf{\Lambda}(\Phi,\Psi)\,. \tag{8}$$

If A has eigenvalues in $\mathbf{\Lambda}(\Phi,\Psi)$, the statement in the lemma is stronger than (8) [12]. For the sake of brevity, only strict FDIs are considered in this paper. The gKYP lemma readily extends to nonstrict inequalities if a regularity condition is imposed [4]. In addition, the result can be generalized to FDIs in which $\left[\begin{smallmatrix}A & B\\ I & 0\end{smallmatrix}\right]$ is replaced by an arbitrary matrix [4].

In [4] Iwasaki and Hara prove the gKYP lemma by showing that for $\mathbf{M}$ defined as

$$\mathbf{M}=\Bigg\{\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^H(\Phi\otimes P+\Psi\otimes Q)\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\;\Bigg|\\ P,Q\in\mathbf{H}_n,\;Q\succ 0\Bigg\}\,.$$

the following two statements are equivalent:

1) If $w\neq 0$ and $w^HMw\geq 0$ for all $M\in\mathbf{M}$ then $w^H\Theta w<0$.
2) There exists $M\in\mathbf{M}$ for which $M+\Theta\prec 0$.

They refer to this result as the *lossless S-procedure*. More generally, they consider cones $\mathbf{M}$ that are admissible (closed, convex, nonempty, and containing no negative definite elements) and show that the lossless S-procedure holds for arbitrary Θ if and only if $\mathbf{M}$ is *rank-one separable*, i.e., the intersection of the dual cone of $\mathbf{M}$ with the positive semidefinite cone is equal to the conic hull of its rank-one elements.

In this paper we show that if the matrices A, B, Θ, and Φ are real, then the matrices P and Q in (7) can be constrained to be real symmetric. In the terminology of [4], this result means that although the set

$$\mathbf{M}_\mathrm{r}=\Bigg\{\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^T(\Phi\otimes P+\Psi\otimes Q)\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\;\Bigg|\\ P,Q\in\mathbf{S}_n,\;Q\succ 0\Bigg\}\,.$$

(with A, B, and Φ real) is not rank-one separable and therefore the lossless S-procedure does not hold for general Θ, it does hold if we restrict Θ to be real.

III. Preliminary Results

The following lemmas will be important in the proof of our main result.

Lemma 1. *Let $\Phi_\mathrm{o},\Psi_\mathrm{o}\in\mathbf{S}_2$ be defined in (4b) with $0\leq\alpha\leq\gamma$ or $\alpha<0<\gamma$. Matrices $X,Y\in\mathbf{C}_{n\times m}$ satisfy*

$$\begin{bmatrix}X & Y\end{bmatrix}(\Phi_\mathrm{o}\otimes I_m)\begin{bmatrix}X^H\\ Y^H\end{bmatrix}=0\,, \tag{9a}$$

$$\begin{bmatrix}X & Y\end{bmatrix}(\Psi_\mathrm{o}\otimes I_m)\begin{bmatrix}X^H\\ Y^H\end{bmatrix}\succeq 0\,, \tag{9b}$$

if and only if they can be factored as

$$X=W\,\mathbf{diag}(s)V^H\,,\qquad Y=W\,\mathbf{diag}(t)V^H\,, \tag{10}$$

where $W\in\mathbf{C}_{n\times m}$, $V\in\mathbf{C}_{m\times m}$, with V unitary, and $s,t\in\mathbf{C}_m$ satisfy $(s_i,t_i)\in\mathbf{\Sigma}(\Phi_\mathrm{o},\Psi_\mathrm{o})$ for $i=1,\ldots,m$.

In the case of $0\leq\alpha\leq\gamma$, the inequality (9b) is redundant and the result follows from [13, lemma 5]. The proof for $\alpha<0<\gamma$ follows from [11, lemma 5]. By means of the congruence transformation (4), lemma 1 is readily extended to other $\Phi,\Psi\in\mathbf{H}_2$.

Corollary 1. *Suppose $\Phi,\Psi\in\mathbf{H}_2$ define a curve $\mathbf{\Lambda}(\Phi,\Psi)$ in the complex plane. Matrices $F,G\in\mathbf{C}_{n\times m}$ satisfy*

$$\begin{bmatrix}F & G\end{bmatrix}(\bar\Phi\otimes I_m)\begin{bmatrix}F^H\\ G^H\end{bmatrix}=0\,, \tag{11a}$$

$$\begin{bmatrix}F & G\end{bmatrix}(\bar\Psi\otimes I_m)\begin{bmatrix}F^H\\ G^H\end{bmatrix}\succeq 0\,, \tag{11b}$$

if and only if they can be factored as

$$F=W\,\mathbf{diag}(\mu)V^H\,,\qquad G=W\,\mathbf{diag}(\nu)V^H\,, \tag{12}$$

where $W\in\mathbf{C}_{n\times m}$, $V\in\mathbf{C}_{m\times m}$, with V unitary, and $\mu,\nu\in\mathbf{C}_m$ satisfy $(\mu_i,\nu_i)\in\mathbf{\Sigma}(\Phi,\Psi)$ for $i=1,\ldots,m$.

This corollary follows from [4, lemma 1]. For our purposes we will need the following corollary of lemma 1.

Corollary 2. *Suppose $\Phi \in \mathbf{S}_2$ and $\Psi \in \mathbf{H}_2$ define a curve $\mathbf{\Lambda}(\Phi, \Psi)$ in the complex plane. Matrices $F, G \in \mathbf{C}_{n\times m}$ satisfy*

$$\begin{bmatrix} F & G \end{bmatrix} (\Phi \otimes I_m) \begin{bmatrix} F^H \\ G^H \end{bmatrix} = 0 \,, \quad \text{(13a)}$$

$$\begin{bmatrix} F & G \end{bmatrix} \big(\bar{\Psi} \otimes \mathbf{diag}(I_{m_1}, 0_{m_2\times m_2}) + \Psi \otimes \mathbf{diag}(0_{m_1\times m_1}, I_{m_2})\big) \begin{bmatrix} F^H \\ G^H \end{bmatrix} \succeq 0 \,, \quad \text{(13b)}$$

where $m_1 + m_2 = m$, if and only if they can be factored as

$$F = W\,\mathbf{diag}(\mu)V^H \,, \qquad G = W\,\mathbf{diag}(\nu)V^H \,, \qquad (14)$$

where $W \in \mathbf{C}_{n\times m}$, $V \in \mathbf{C}_{m\times m}$, with V unitary, and $\mu, \nu \in \mathbf{C}_m$ satisfy $(\mu_i, \nu_i) \in \mathbf{\Sigma}(\Phi, \Psi) \cup \mathbf{\Sigma}(\Phi, \bar{\Psi})$ for $i = 1, \dots, m$.

Proof: Define the matrix $Z \in \mathbf{C}_{2m\times 2m}$ as

$$\begin{aligned} Z &= T^T \otimes \mathbf{diag}(I_{m_1}, 0_{m_2\times m_2}) + T^H \otimes \mathbf{diag}(0_{m_1\times m_1}, I_{m_2}) \\ &= \left[\begin{array}{cc|cc} T_{11}I_{m_1} & 0 & T_{21}I_{m_1} & 0 \\ 0 & \bar{T}_{11}I_{m_2} & 0 & \bar{T}_{21}I_{m_2} \\ \hline T_{12}I_{m_1} & 0 & T_{22}I_{m_1} & 0 \\ 0 & \bar{T}_{12}I_{m_2} & 0 & \bar{T}_{22}I_{m_2} \end{array}\right] \end{aligned}$$

and the matrices $X, Y \in \mathbf{C}_{n\times m}$ as

$$\begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} F & G \end{bmatrix} Z \,. \qquad (15)$$

In other words, if we partition X, Y, F, G as

$$\begin{aligned} X &= \begin{bmatrix} X_1 & X_2 \end{bmatrix}, & Y &= \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}, \\ F &= \begin{bmatrix} F_1 & F_2 \end{bmatrix}, & G &= \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \end{aligned}$$

with X_i, Y_i, F_i, $G_i \in \mathbf{C}_{n\times m_i}$, then

$$\begin{aligned} \begin{bmatrix} X_1 & Y_1 \end{bmatrix} &= \begin{bmatrix} F_1 & G_1 \end{bmatrix} (T^T \otimes I_{m_1}), \\ \begin{bmatrix} X_2 & Y_2 \end{bmatrix} &= \begin{bmatrix} F_2 & G_2 \end{bmatrix} (T^H \otimes I_{m_2}). \end{aligned}$$

The inequality (13b) implies that

$$\begin{aligned} &\begin{bmatrix} X & Y \end{bmatrix} (\Psi_o \otimes I_m) \begin{bmatrix} X^H \\ Y^H \end{bmatrix} \\ &= \begin{bmatrix} X_1 & Y_1 \end{bmatrix} (\Psi_o \otimes I_{m_1}) \begin{bmatrix} X_1^H \\ Y_1^H \end{bmatrix} + \begin{bmatrix} X_2 & Y_2 \end{bmatrix} (\Psi_o \otimes I_{m_2}) \begin{bmatrix} X_2^H \\ Y_2^H \end{bmatrix} \\ &= \begin{bmatrix} F_1 & G_1 \end{bmatrix} \big((T^T \Psi_o \bar{T}) \otimes I_{m_1}\big) \begin{bmatrix} F_1^H \\ G_1^H \end{bmatrix} + \begin{bmatrix} F_2 & G_2 \end{bmatrix} \big((T^H \Psi_o T) \otimes I_{m_2}\big) \begin{bmatrix} F_2^H \\ G_2^H \end{bmatrix} \\ &= \begin{bmatrix} F_1 & G_1 \end{bmatrix} (\bar{\Psi} \otimes I_{m_1}) \begin{bmatrix} F_1^H \\ G_1^H \end{bmatrix} + \begin{bmatrix} F_2 & G_2 \end{bmatrix} (\Psi \otimes I_{m_2}) \begin{bmatrix} F_2^H \\ G_2^H \end{bmatrix} \\ &= \begin{bmatrix} F & G \end{bmatrix} \big(\bar{\Psi} \otimes \mathbf{diag}(I_{m_1}, 0_{m_2\times m_2}) + \Psi \otimes \mathbf{diag}(0_{m_1\times m_1}, I_{m_2})\big) \begin{bmatrix} F^H \\ G^H \end{bmatrix} \\ &\succeq 0 \,. \end{aligned}$$

Similarly, the equality (13a) and the fact that Φ is real imply that

$$\begin{aligned} &\begin{bmatrix} X & Y \end{bmatrix} (\Phi_o \otimes I_m) \begin{bmatrix} X^H \\ Y^H \end{bmatrix} \\ &= \begin{bmatrix} F_1 & G_1 \end{bmatrix} (\bar{\Phi} \otimes I_{m_1}) \begin{bmatrix} F_1^H \\ G_1^H \end{bmatrix} + \begin{bmatrix} F_2 & G_2 \end{bmatrix} (\Phi \otimes I_{m_2}) \begin{bmatrix} F_2^H \\ G_2^H \end{bmatrix} \\ &= \begin{bmatrix} F & G \end{bmatrix} (\Phi \otimes I_m) \begin{bmatrix} F^H \\ G^H \end{bmatrix} \\ &= 0 \,. \end{aligned}$$

Applying lemma 1 we conclude that X and Y can be factored as

$$X = W_1\,\mathbf{diag}(s)\widetilde{V}^H \,, \qquad Y = W_1\,\mathbf{diag}(t)\widetilde{V}^H \,,$$

where $W_1 \in \mathbf{C}_{n\times m}$, $\widetilde{V} \in \mathbf{C}_{m\times m}$, with $\widetilde{V}$ unitary, and $s, t \in \mathbf{C}_m$ satisfy $(s_i, t_i) \in \mathbf{\Sigma}(\Phi_o, \Psi_o)$ for $i = 1, \dots, m$. Transforming X and Y back to the original matrices F and G yields

$$F = W_1 \Delta \,, \qquad G = W_1 \Gamma \,, \qquad (16)$$

where $\Delta, \Gamma \in \mathbf{C}_{m\times m}$ equal

$$\begin{bmatrix} \Delta & \Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(s) & \mathbf{diag}(t) \end{bmatrix} \begin{bmatrix} \widetilde{V}^H & 0 \\ 0 & \widetilde{V}^H \end{bmatrix} Z^{-1} \,. \qquad (17)$$

It is readily verified from the property $(s_i, t_i) \in \mathbf{\Sigma}(\Phi_o, \Psi_o)$ and the definition of Z that Δ, Γ satisfy

$$\begin{bmatrix} \Delta & \Gamma \end{bmatrix} (\Phi \otimes I_m) \begin{bmatrix} \Delta^H \\ \Gamma^H \end{bmatrix} = 0 \,, \quad \text{(18a)}$$

$$\begin{bmatrix} \Delta & \Gamma \end{bmatrix} \big(\bar{\Psi} \otimes \mathbf{diag}(I_{m_1}, 0_{m_2\times m_2}) + \Psi \otimes \mathbf{diag}(0_{m_1\times m_1}, I_{m_2})\big) \begin{bmatrix} \Delta^H \\ \Gamma^H \end{bmatrix} \succeq 0 \,. \quad \text{(18b)}$$

In addition, the intersection of the nullspaces of the matrices Δ^H and Γ^H does not contain nonzero elements: since the two right-most matrices in (17) are nonsingular, we have

$$v^H \begin{bmatrix} \Delta & \Gamma \end{bmatrix} = 0 \quad \Leftrightarrow \quad v^H \begin{bmatrix} \mathbf{diag}(s) & \mathbf{diag}(t) \end{bmatrix} = 0,$$

and the right-hand side implies that $v = 0$ because $(s_i, t_i) \neq (0, 0)$ for $i = 1, \dots, m$.

To derive the factorization (14), we apply corollary 1 (with $\Psi = 0$) to the equality (18a). This yields a factorization

$$\Delta = W_2\,\mathbf{diag}(\mu)V^H \,, \qquad \Gamma = W_2\,\mathbf{diag}(\nu)V^H \qquad (19)$$

for some $W_2 \in \mathbf{C}_{m\times m}$, unitary $V \in \mathbf{C}_{m\times m}$, and vectors $\mu, \nu \in \mathbf{C}_m$ with $(\mu_i, \nu_i) \neq 0$ and

$$\begin{bmatrix} \mu_i \\ \nu_i \end{bmatrix}^H \Phi \begin{bmatrix} \mu_i \\ \nu_i \end{bmatrix} = 0 \,, \qquad (20)$$

for $i = 1, \dots, m$. Since the intersection of the nullspaces of Δ^H and Γ^H has dimension zero, the matrix W_2 is invertible. Now let e_i denote the ith unit vector in $\mathbf{R}_m$ and v_i denote the ith column of V. Then, multiplying the inequality (18b) on the left with $e_i^T W_2^{-1}$ and on the right with $W_2^{-H} e_i$ yields

$$\|v_{i,1}\|^2 \begin{bmatrix} \mu_i \\ \nu_i \end{bmatrix}^H \bar{\Psi} \begin{bmatrix} \mu_i \\ \nu_i \end{bmatrix} + \|v_{i,2}\|^2 \begin{bmatrix} \mu_i \\ \nu_i \end{bmatrix}^H \Psi \begin{bmatrix} \mu_i \\ \nu_i \end{bmatrix} \geq 0 \,,$$

where $v_i = \begin{bmatrix} v_{i,1}^T & v_{i,2}^T \end{bmatrix}^T$ is a partitioning of v_i with $v_{i,1} \in \mathbf{C}_{m_1}$, $v_{i,2} \in \mathbf{C}_{m_2}$. Hence, since $v_i \neq 0$, (μ_i, ν_i) satisfy

$$\begin{bmatrix}\mu_i \\ \nu_i\end{bmatrix}^H \Psi \begin{bmatrix}\mu_i \\ \nu_i\end{bmatrix} \geq 0 \qquad \text{or} \qquad \begin{bmatrix}\mu_i \\ \nu_i\end{bmatrix}^H \bar{\Psi} \begin{bmatrix}\mu_i \\ \nu_i\end{bmatrix} \geq 0 \,,$$

in addition to (20). Consequently, $(\mu_i, \nu_i) \in \mathbf{\Sigma}(\Phi, \Psi) \cup \mathbf{\Sigma}(\Phi, \bar{\Psi})$. Combining (16) and (19) and defining $W = W_1 W_2$ yields the factorization (14). ∎

IV. Generalized KYP Lemma with Real Data

In this section we consider the generalized KYP lemma for an FDI with real data matrices $A \in \mathbf{R}_{n\times n}$, $B \in \mathbf{R}_{n\times m}$, and $\Theta \in \mathbf{S}_{n+m}$. In addition, $\Phi \in \mathbf{S}_2$ is assumed, which covers the most common instances $\Phi_{\rm c}$ and $\Phi_{\rm d}$ (5). With real matrices A, B and Θ, and $\Phi \in \mathbf{S}_2$, the inequality (8) for a given λ implies that the inequality also holds for $\bar{\lambda}$. Hence, each curve $\mathbf{\Lambda}(\Phi, \Psi)$ may be extended to

$$\widehat{\mathbf{\Lambda}}(\Phi, \Psi) = \mathbf{\Lambda}(\Phi, \Psi) \cup \mathbf{\Lambda}(\Phi, \bar{\Psi}) \,,$$

without introducing conservatism. If $\Psi \in \mathbf{S}_2$, $\widehat{\mathbf{\Lambda}}(\Phi, \Psi) = \mathbf{\Lambda}(\Phi, \Psi)$ and it is readily observed that the existence of real symmetric matrices P and Q that satisfy (7) is necessary and sufficient for the FDI (6) to hold. On the other hand, if Ψ is complex, $\mathbf{\Lambda}(\Phi, \Psi) \subsetneq \widehat{\mathbf{\Lambda}}(\Phi, \Psi)$ and at first sight, the LMI condition (7) with $P, Q \in \mathbf{S}_n$ seems only a sufficient condition for (6). However, as we will show, even with Ψ Hermitian, the matrix variables P and Q can be restricted to be real symmetric without loss of generality.

Theorem 2. *Let $A \in \mathbf{R}_{n\times n}$, $B \in \mathbf{R}_{n\times m}$, and $\Theta \in \mathbf{S}_{n+m}$. Suppose $\Phi \in \mathbf{S}_2$ and $\Psi \in \mathbf{H}_2$ define a curve $\mathbf{\Lambda}(\Phi, \Psi)$ in the complex plane. Then the following two statements are equivalent.*

1) *If $\lambda \in \widehat{\mathbf{\Lambda}}(\Phi, \Psi)$, then*

$$\begin{bmatrix}u \\ v\end{bmatrix}^H \Theta \begin{bmatrix}u \\ v\end{bmatrix} < 0 \qquad (21)$$

for all nonzero $(u, v) \in N_\lambda(A, B)$.

2) *There exist $P, Q \in \mathbf{S}_n$ that satisfy*

$$Q \succ 0, \quad \begin{bmatrix}A & B \\ I & 0\end{bmatrix}^T (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Theta \prec 0 \,. \qquad (22)$$

Proof: The proof that statement 2 implies statement 1 follows from the gKYP lemma (theorem 1). We prove the converse by contradiction using a theorem of alternatives [12, Theorem 1] [14, Theorem 1.3]: from the infeasibility certificate of (22) we construct a $\lambda \in \widehat{\mathbf{\Lambda}}(\Phi, \Psi)$ and nonzero $(u, v) \in N_\lambda(A, B)$ that violate (21). Suppose (22) is infeasible. Then there exists a nonzero positive semidefinite matrix $Z \in \mathbf{H}_{n+m}$ such that $\mathbf{tr}(\Theta Z) \geq 0$ and

$$\Re(\Phi_{11} S_{11} + \Phi_{21} S_{12} + \Phi_{12} S_{21} + \Phi_{22} S_{22}) = 0, \qquad (23a)$$

$$\Re(\Psi_{11} S_{11} + \Psi_{21} S_{12} + \Psi_{12} S_{21} + \Psi_{22} S_{22}) \succeq 0, \qquad (23b)$$

where

$$S = \begin{bmatrix}S_{11} & S_{12} \\ S_{21} & S_{22}\end{bmatrix} = \begin{bmatrix}A & B \\ I & 0\end{bmatrix} Z \begin{bmatrix}A & B \\ I & 0\end{bmatrix}^T .$$

Since Z is nonzero and positive semidefinite it can be factored as

$$Z = \begin{bmatrix}X \\ Y\end{bmatrix} \begin{bmatrix}X \\ Y\end{bmatrix}^H$$

with $X \in \mathbf{C}_{n\times r}$, $Y \in \mathbf{C}_{m\times r}$ and $r \geq 1$. The equality and inequality in (23) can then be written in the form (13) with $m_1 = m_2 = r$ and

$$F = \begin{bmatrix} AX + BY & \overline{AX + BY} \end{bmatrix} , \qquad G = \begin{bmatrix} X & \bar{X} \end{bmatrix} .$$

Application of corollary 2 and the property that A and B are real gives the factorization

$$\begin{bmatrix}A & B\end{bmatrix} \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix} = W \, \mathbf{diag}(\mu) V^H, \qquad (24a)$$

$$\begin{bmatrix}X & \bar{X}\end{bmatrix} = W \, \mathbf{diag}(\nu) V^H \,, \qquad (24b)$$

where V is unitary and $(\mu_i, \nu_i) \in \mathbf{\Sigma}(\Phi, \Psi) \cup \mathbf{\Sigma}(\Phi, \bar{\Psi})$ for $i = 1, \ldots, 2r$.

Next we note that

$$\Re(Z) = \frac{1}{2} \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix} \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix}^H = \frac{1}{2} \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix} V V^H \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix}^H$$

and $\Re(Z) \neq 0$ because $Z \gneq 0$. Also

$$\mathbf{tr}(\Theta Z) = \frac{1}{2} \mathbf{tr} \left(V^H \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix}^H \Theta \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix} V \right) \geq 0.$$

Therefore there is at least one column v_i of V for which

$$\begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix} v_i \neq 0, \qquad v_i^H \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix}^H \Theta \begin{bmatrix}X & \bar{X} \\ Y & \bar{Y}\end{bmatrix} v_i \geq 0.$$

Defining

$$u = \begin{bmatrix}X & \bar{X}\end{bmatrix} v_i, \qquad v = \begin{bmatrix}Y & \bar{Y}\end{bmatrix} v_i$$

gives a vector $(u, v) \neq 0$ with

$$\begin{bmatrix}u \\ v\end{bmatrix}^H \Theta \begin{bmatrix}u \\ v\end{bmatrix} \geq 0, \quad Au + Bv = w_i \mu_i, \quad u = w_i \nu_i,$$

where w_i is the ith column of W and $(\mu_i, \nu_i) \in \mathbf{\Sigma}(\Phi, \Psi) \cup \mathbf{\Sigma}(\Phi, \bar{\Psi})$. If we define $\lambda = \mu_i/\nu_i$ if $\nu_i \neq 0$ and $\lambda = \infty$ otherwise, then $(u, v) \in N_\lambda(A, B)$ and $\lambda \in \widehat{\mathbf{\Lambda}}(\Phi, \Psi)$, and we reach a contradiction with statement 1. ∎

V. Conclusion

In many applications, the gKYP lemma is applied to a continuous-time or discrete-time frequency-domain inequality that involves real data matrices. It is shown here that for such applications, the matrix variables in the gKYP-LMI may always be constrained to be real symmetric instead of complex Hermitian without introducing conservatism, even if the considered frequency domain is asymmetric around the real axis. This has computational advantages because it reduces the number of variables in the gKYP-LMI.

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