Convex Optimization: Modeling and Algorithms

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Introduction

- mathematical optimization
- linear and convex optimization
- recent history

Mathematical optimization

$$\begin{array}{ll}\text{minimize} & f_0(x_1,\ldots,x_n)\\ \text{subject to} & f_1(x_1,\ldots,x_n)\le 0\\ & \cdots\\ & f_m(x_1,\ldots,x_n)\le 0\end{array}$$

- a mathematical model of a decision, design, or estimation problem
- finding a global solution is generally intractable
- even simple looking nonlinear optimization problems can be very hard

The famous exception: Linear programming

$$\begin{array}{ll} \text{minimize} & c_1x_1 + \cdots c_2x_2 \\ \text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & \ldots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{array}$$

- widely used since Dantzig introduced the simplex algorithm in 1948
- since 1950s, many applications in operations research, network optimization, finance, engineering, combinatorial optimization, . . .
- extensive theory (optimality conditions, sensitivity analysis, . . .)
- there exist very efficient algorithms for solving linear programs

Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex: for $0 \leq \theta \leq 1$

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

- can be solved globally, with similar (polynomial-time) complexity as LPs
- surprisingly many problems can be solved via convex optimization
- provides tractable heuristics and relaxations for non-convex problems

History

- 1940s: linear programming

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

- 1950s: quadratic programming
- 1960s: geometric programming
- 1990s: semidefinite programming, second-order cone programming, quadratically constrained quadratic programming, robust optimization, sum-of-squares programming, . . .

New applications since 1990

- linear matrix inequality techniques in control
- support vector machine training via quadratic programming
- semidefinite programming relaxations in combinatorial optimization
- circuit design via geometric programming
- ℓ_1-norm optimization for sparse signal reconstruction
- applications in structural optimization, statistics, signal processing, communications, image processing, computer vision, quantum information theory, finance, power distribution, . . .

Advances in convex optimization algorithms

interior-point methods

- 1984 (Karmarkar): first practical polynomial-time algorithm for LP
- 1984-1990: efficient implementations for large-scale LPs
- around 1990 (Nesterov & Nemirovski): polynomial-time interior-point methods for nonlinear convex programming
- since 1990: extensions and high-quality software packages

first-order algorithms

- fast gradient methods, based on Nesterov's methods from 1980s
- extend to certain nondifferentiable or constrained problems
- multiplier methods for large-scale and distributed optimization

Overview

1. Basic theory and convex modeling

- convex sets and functions
- common problem classes and applications

2. Interior-point methods for conic optimization

- conic optimization
- barrier methods
- symmetric primal-dual methods

3. First-order methods

- (proximal) gradient algorithms
- dual techniques and multiplier methods

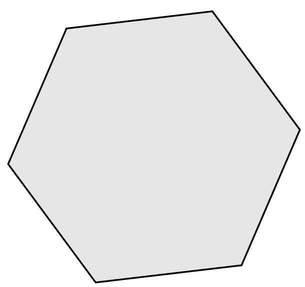
Convex sets and functions

- convex sets
- convex functions
- operations that preserve convexity

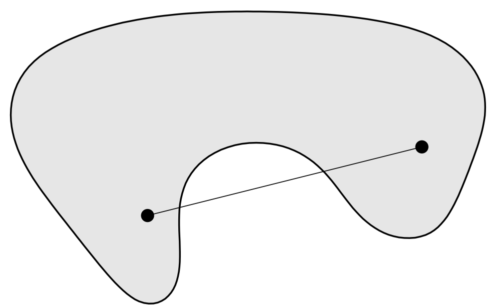
Convex set

contains the line segment between any two points in the set

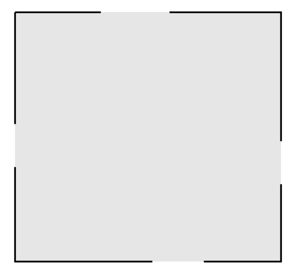
$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta) x_2 \in C$$



convex



not convex



not convex

Basic examples

affine set: solution set of linear equations $Ax = b$

halfspace: solution of one linear inequality $a^T x \leq b$ $(a \neq 0)$

polyhedron: solution of finitely many linear inequalities $Ax \leq b$

ellipsoid: solution of positive definite quadratic inquality

$$(x - x_{\rm c})^T A(x - x_{\rm c}) \leq 1 \qquad (A \text{ positive definite})$$

norm ball: solution of $\|x\| \leq R$ (for any norm)

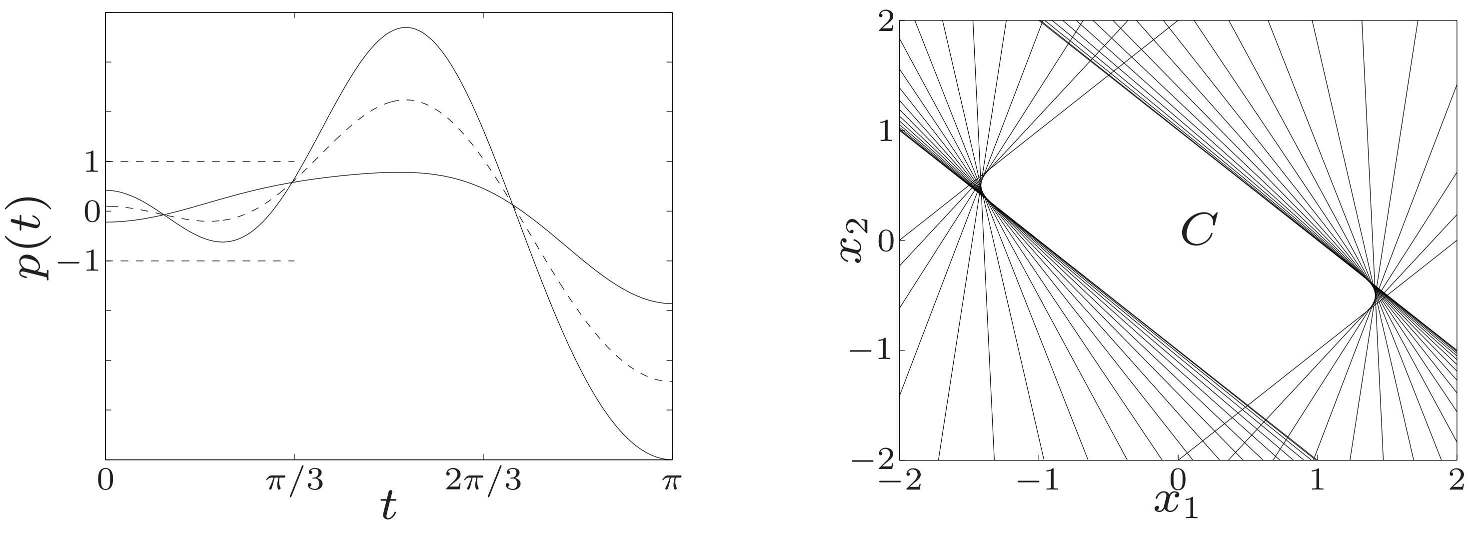
positive semidefinite cone: $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$

the **intersection** of any number of convex sets is convex

Example of intersection property

$$C = \{x \in \mathbf{R}^n \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_n \cos nt$



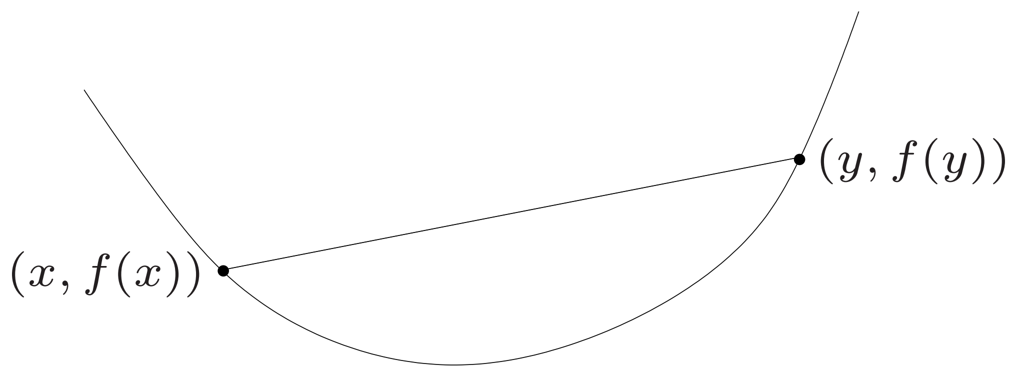
C is intersection of infinitely many halfspaces, hence convex

Convex function

domain $\mathbf{dom}\, f$ is a convex set and Jensen's inequality holds:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$



f is concave if $-f$ is convex

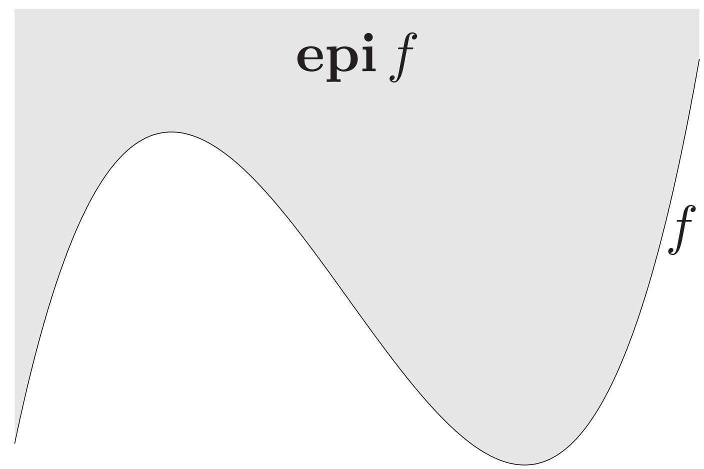
Examples

- linear and affine functions are convex and concave
- $\exp x$, $-\log x$, $x \log x$ are convex
- x^α is convex for $x > 0$ and $\alpha \geq 1$ or $\alpha \leq 0$; $|x|^\alpha$ is convex for $\alpha \geq 1$
- norms are convex
- quadratic-over-linear function $x^T x/t$ is convex in x, t for $t > 0$
- geometric mean $(x_1 x_2 \cdots x_n)^{1/n}$ is concave for $x \geq 0$
- $\log \det X$ is concave on set of positive definite matrices
- $\log(e^{x_1} + \cdots e^{x_n})$ is convex

Epigraph and sublevel set

epigraph: $\textbf{epi}\, f = \{(x, t) \mid x \in \textbf{dom}\, f,\ f(x) \leq t\}$

a function is convex if and only its epigraph is a convex set



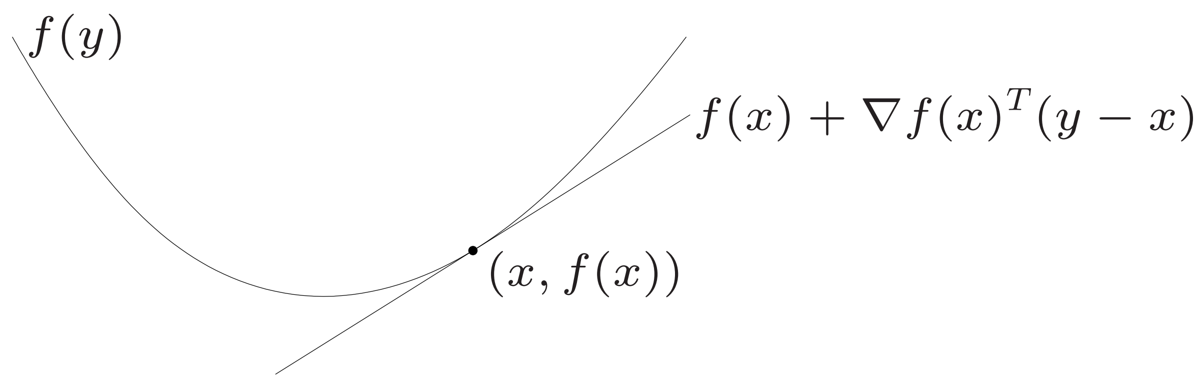
sublevel sets: $C_\alpha = \{x \in \textbf{dom}\, f \mid f(x) \leq \alpha\}$

the sublevel sets of a convex function are convex (converse is false)

Differentiable convex functions

differentiable f is convex if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$



twice differentiable f is convex if and only if $\mathbf{dom}\, f$ is convex and

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \mathbf{dom}\, f$$

Establishing convexity of a function

1. verify definition

2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

3. show that f is obtained from simple convex functions by operations that preserve convexity

    - nonnegative weighted sum
    - composition with affine function
    - pointwise maximum and supremum
    - minimization
    - composition
    - perspective

Positive weighted sum & composition with affine function

nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax + b)$ is convex if f is convex

examples

- logarithmic barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is convex if $f_1, \ldots, f_m$ are convex

example: sum of r largest components of $x \in \mathbf{R}^n$

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

is convex ($x_{[i]}$ is ith largest component of x)

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

Pointwise supremum

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$

examples

- maximum eigenvalue of symmetric matrix

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

- support function of a set C

$$S_C(x) = \sup_{y \in C} y^T x$$

Minimization

$$h(x) = \inf_{y \in C} f(x, y)$$

is convex if $f(x, y)$ is convex in (x, y) and C is a convex set

examples

- distance to a convex set C: $h(x) = \inf_{y \in C} \|x - y\|$
- optimal value of linear program as function of righthand side

$$h(x) = \inf_{y : Ay \leq x} c^T y$$

  follows by taking

$$f(x, y) = c^T y, \qquad \mathbf{dom}\, f = \{(x, y) \mid Ay \leq x\}$$

Composition

composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

f is convex if

$$\begin{array}{l} g \text{ convex, } h \text{ convex and nondecreasing} \\ g \text{ concave, } h \text{ convex and nonincreasing} \end{array}$$

(if we assign $h(x) = \infty$ for $x \in \mathbf{dom}\, h$)

examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Vector composition

composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$:

$$f(x) = h(g(x)) = h\left(g_1(x), g_2(x), \ldots, g_k(x)\right)$$

f is convex if

g_i convex, h convex and nondecreasing in each argument
g_i concave, h convex and nonincreasing in each argument

(if we assign $h(x) = \infty$ for $x \in \mathbf{dom}\, h$)

example

$\log \sum_{i=1}^{m} \exp g_i(x)$ is convex if g_i are convex

Perspective

the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x,t) = tf(x/t)$$

g is convex if f is convex on $\mathbf{dom}\, g = \{(x,t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$

examples

- perspective of $f(x) = x^T x$ is quadratic-over-linear function

$$g(x,t) = \frac{x^T x}{t}$$

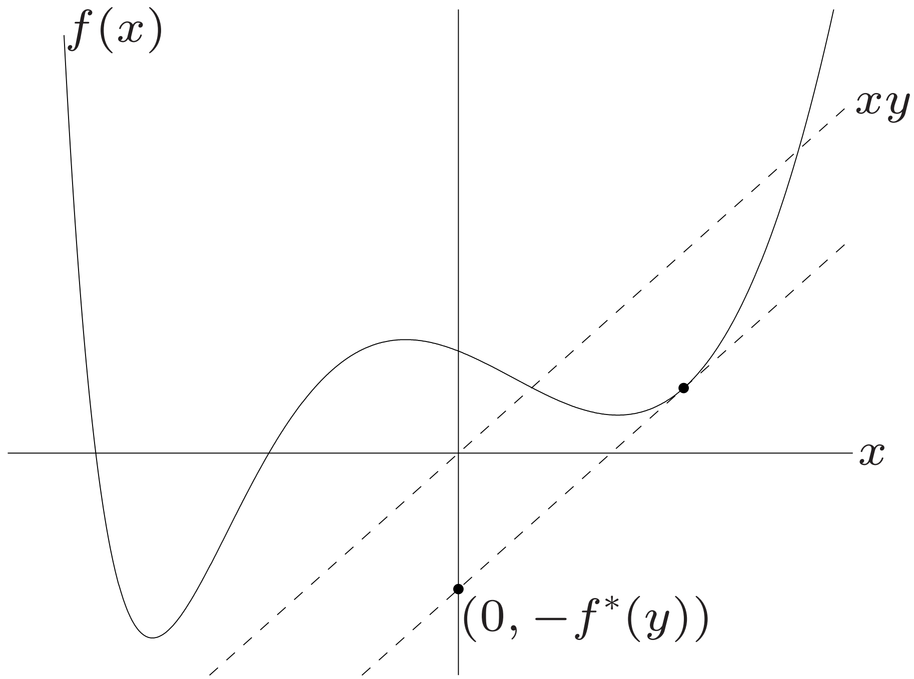
- perspective of negative logarithm $f(x) = -\log x$ is relative entropy

$$g(x,t) = t \log t - t \log x$$

Conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



f^* is convex (even if f is not)

Examples

convex quadratic function ($Q \succ 0$)

$$f(x) = \frac{1}{2}x^T Q x \qquad\qquad f^*(y) = \frac{1}{2}y^T Q^{-1} y$$

negative entropy

$$f(x) = \sum_{i=1}^n x_i \log x_i \qquad\qquad f^*(y) = \sum_{i=1}^n e^{y_i} - 1$$

norm

$$f(x) = \|x\| \qquad\qquad f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

indicator function (C convex)

$$f(x) = I_C(x) = \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise} \end{cases} \qquad\qquad f^*(y) = \sup_{x \in C} y^T x$$

Convex optimization problems

- linear programming
- quadratic programming
- geometric programming
- second-order cone programming
- semidefinite programming

Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

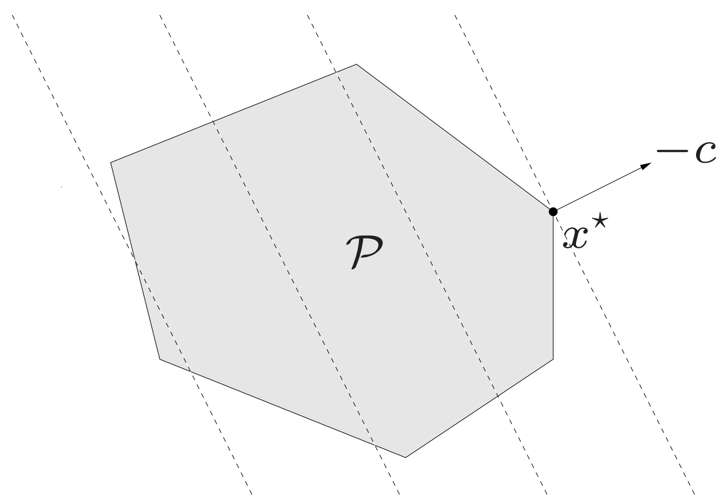
f_0, f_1, . . . , f_m are convex functions

- feasible set is convex
- locally optimal points are globally optimal
- tractable, in theory and practice

Linear program (LP)

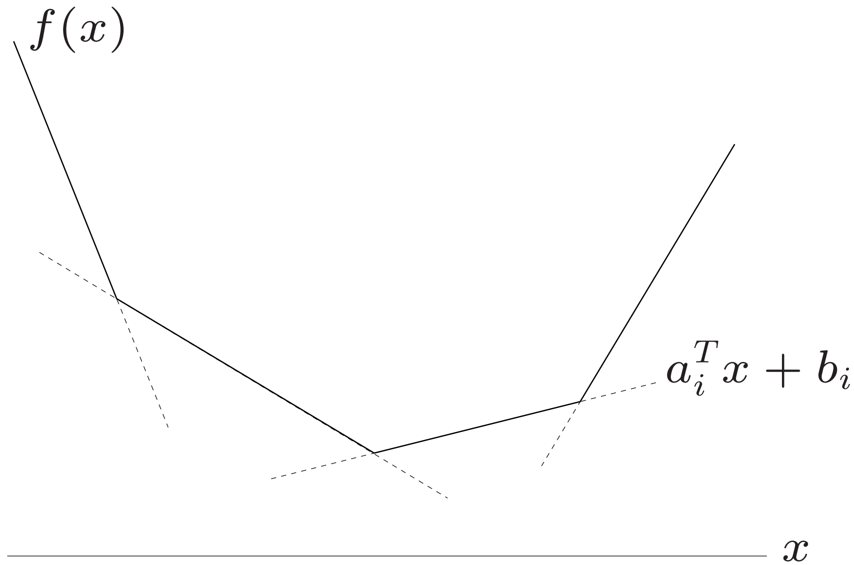
$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- inequality is componentwise vector inequality
- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Piecewise-linear minimization

$$\text{minimize} \quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$



equivalent linear program

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

an LP with variables x, $t \in \mathbf{R}$

ℓ_1-Norm and ℓ_∞-norm minimization

ℓ_1-norm approximation and equivalent LP ($\|y\|_1 = \sum_k |y_k|$)

$$\text{minimize} \quad \|Ax - b\|_1 \qquad\qquad \begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} y_i \\ \text{subject to} & -y \preceq Ax - b \preceq y \end{array}$$

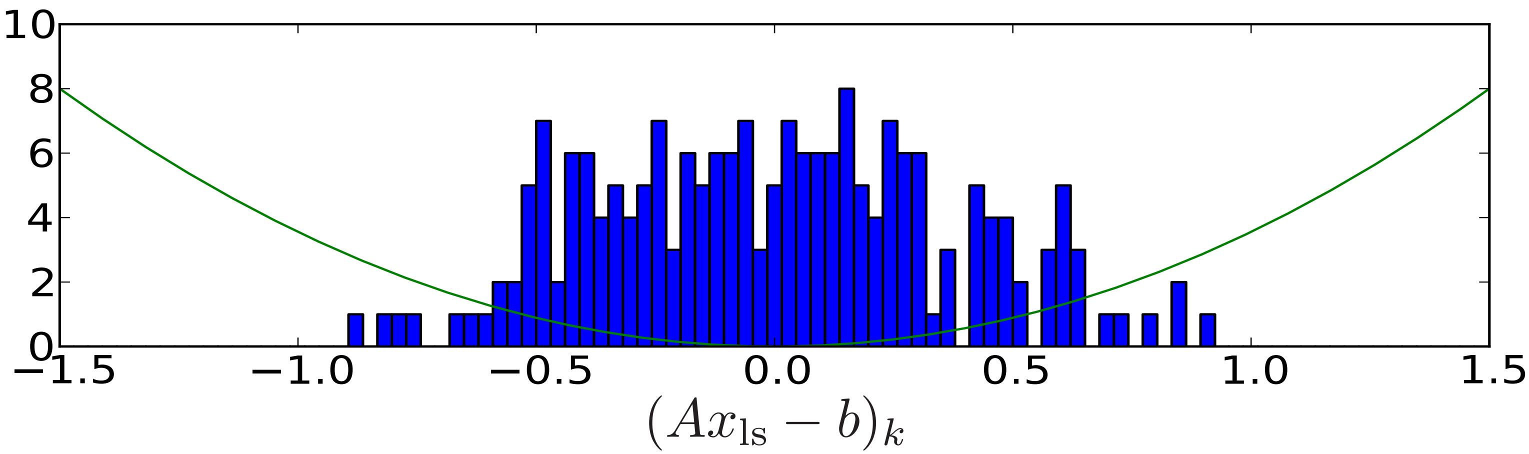
ℓ_∞-norm approximation ($\|y\|_\infty = \max_k |y_k|$)

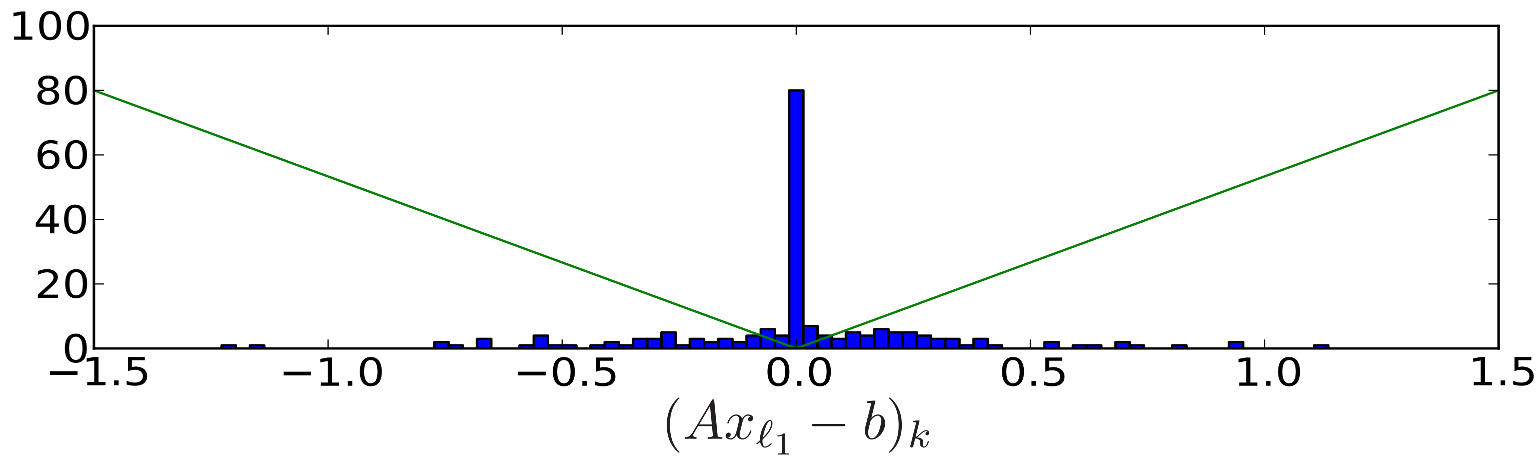
$$\text{minimize} \quad \|Ax - b\|_\infty \qquad\qquad \begin{array}{ll} \text{minimize} & y \\ \text{subject to} & -y\mathbf{1} \preceq Ax - b \preceq y\mathbf{1} \end{array}$$

($\mathbf{1}$ is vector of ones)

example: histograms of residuals $Ax - b$ (with A is 200×80) for

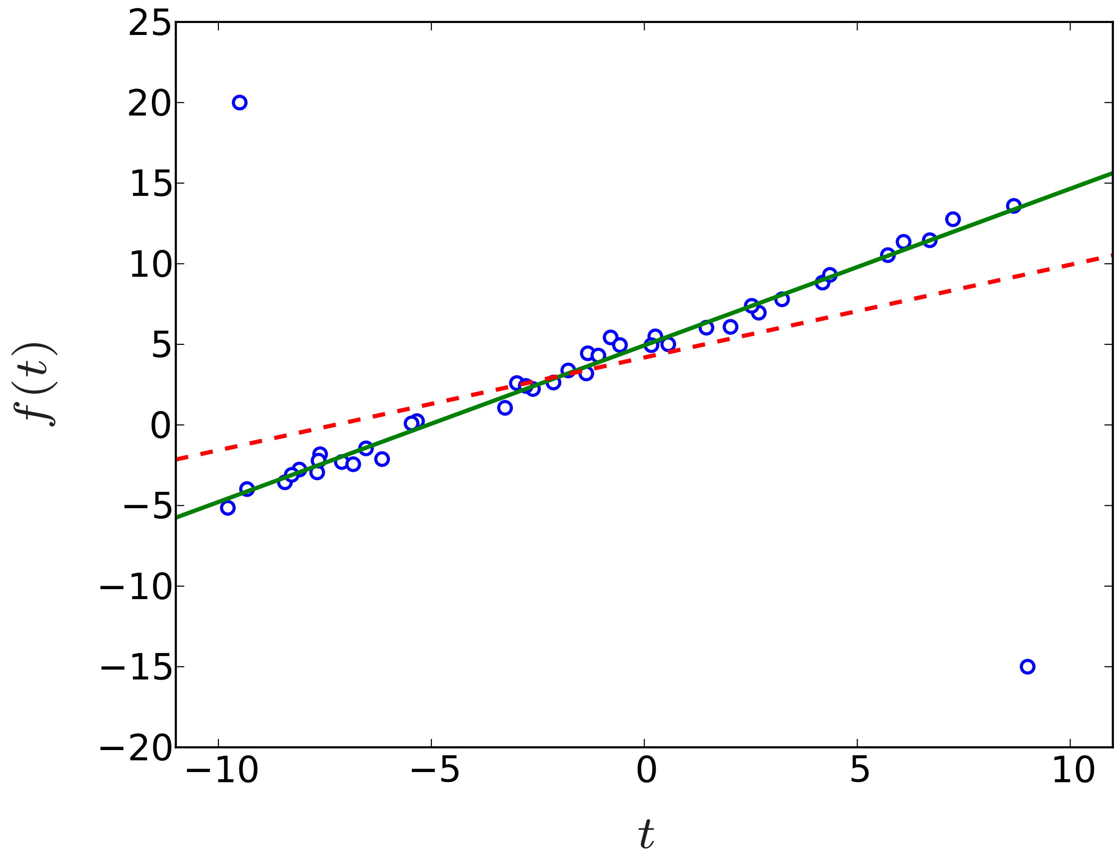
$$x_{\rm ls} = \operatorname{argmin} \|Ax - b\|_2, \qquad x_{\ell_1} = \operatorname{argmin} \|Ax - b\|_1$$





1-norm distribution is wider with a high peak at zero

Robust regression

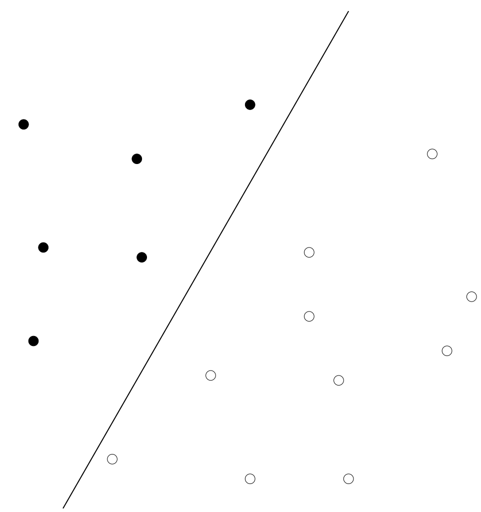


- 42 points t_i, y_i (circles), including two outliers
- function $f(t) = \alpha + \beta t$ fitted using 2-norm (dashed) and 1-norm

Linear discrimination

- given a set of points $\{x_1, \ldots, x_N\}$ with binary labels $s_i \in \{-1, 1\}$
- find hyperplane $a^T x + b = 0$ that strictly separates the two classes

$$\begin{array}{ll} a^T x_i + b > 0 & \text{if } s_i = 1 \\ a^T x_i + b < 0 & \text{if } s_i = -1 \end{array}$$

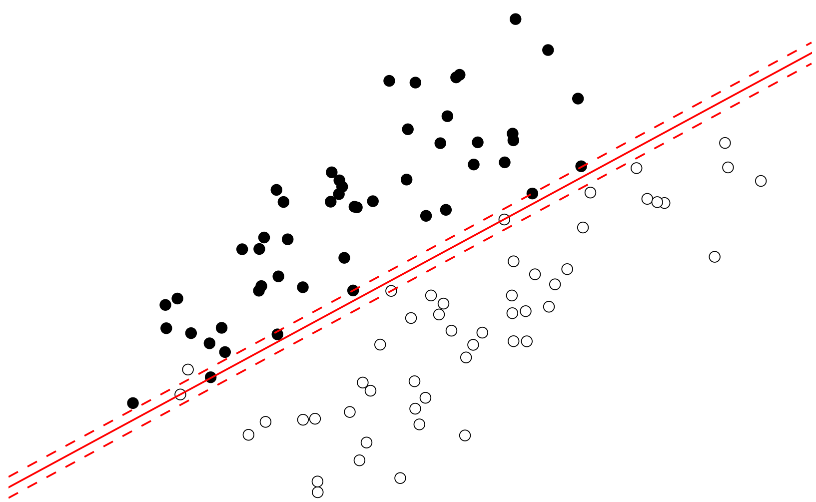


homogeneous in a, b, hence equivalent to the linear inequalities (in a, b)

$$s_i(a^T x_i + b) \geq 1, \quad i = 1, \ldots, N$$

Approximate linear separation of non-separable sets

$$\text{minimize} \quad \sum_{i=1}^{N} \max\{0, 1 - s_i(a^T x_i + b)\}$$

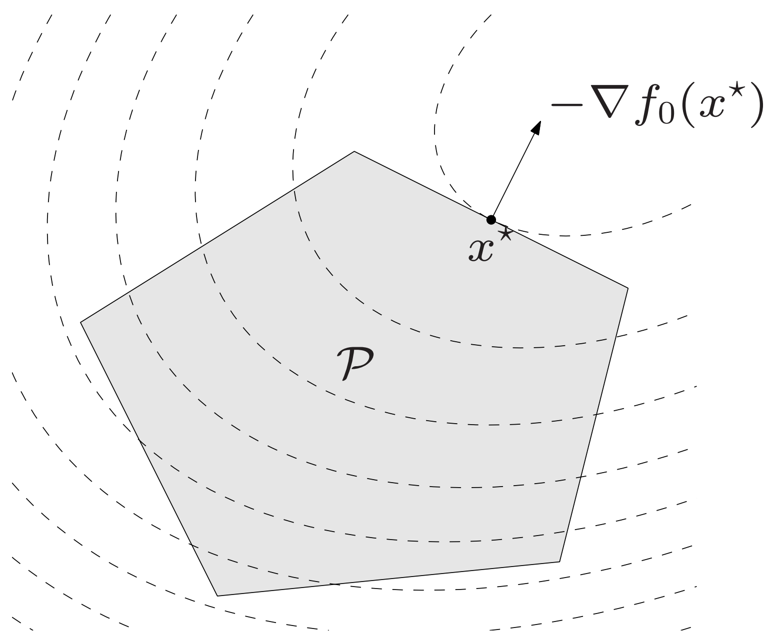


- a piecewise-linear minimization problem in a, b; equivalent to an LP
- can be interpreted as a heuristic for minimizing #misclassified points

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h
\end{array}
$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Linear program with random cost

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \end{array}$$

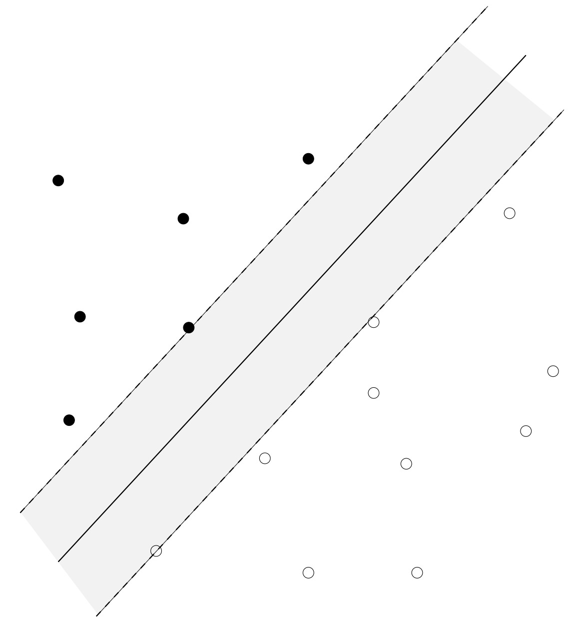
- c is random vector with mean $\bar{c}$ and covariance Σ
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$

expected cost-variance trade-off

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) = \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preceq h \end{array}$$

$\gamma > 0$ is risk aversion parameter

Robust linear discrimination



$$\begin{aligned}\mathcal{H}_1 &= \{z \mid a^T z + b = 1\} \\ \mathcal{H}_{-1} &= \{z \mid a^T z + b = -1\}\end{aligned}$$

distance between hyperplanes is $2/\|a\|_2$

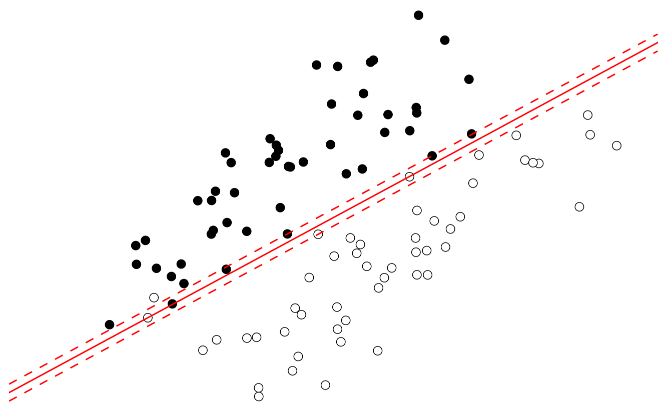
to separate two sets of points by maximum margin,

$$\begin{array}{ll}\text{minimize} & \|a\|_2^2 = a^T a \\ \text{subject to} & s_i(a^T x_i + b) \geq 1, \quad i = 1, \ldots, N\end{array}$$

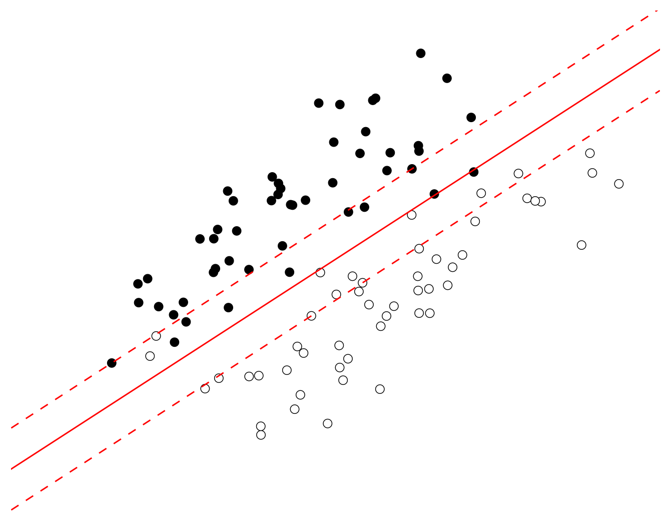
a quadratic program in a, b

Support vector classifier

$$\text{minimize} \quad \gamma\|a\|_2^2 + \sum_{i=1}^{N} \max\{0, 1 - s_i(a^T x_i + b)\}$$



$\gamma = 0$



$\gamma = 10$

equivalent to a quadratic program

Kernel formulation

$$\text{minimize} \quad f(Xa) + \|a\|_2^2$$

- variables $a \in \mathbf{R}^n$
- $X \in \mathbf{R}^{N \times n}$ with $N \leq n$ and rank N

change of variables

$$y = Xa, \qquad a = X^T(XX^T)^{-1}y$$

- a is minimum-norm solution of $Xa = y$
- gives convex problem with N variables y

$$\text{minimize} \quad f(y) + y^T Q^{-1} y$$

$Q = XX^T$ is kernel matrix

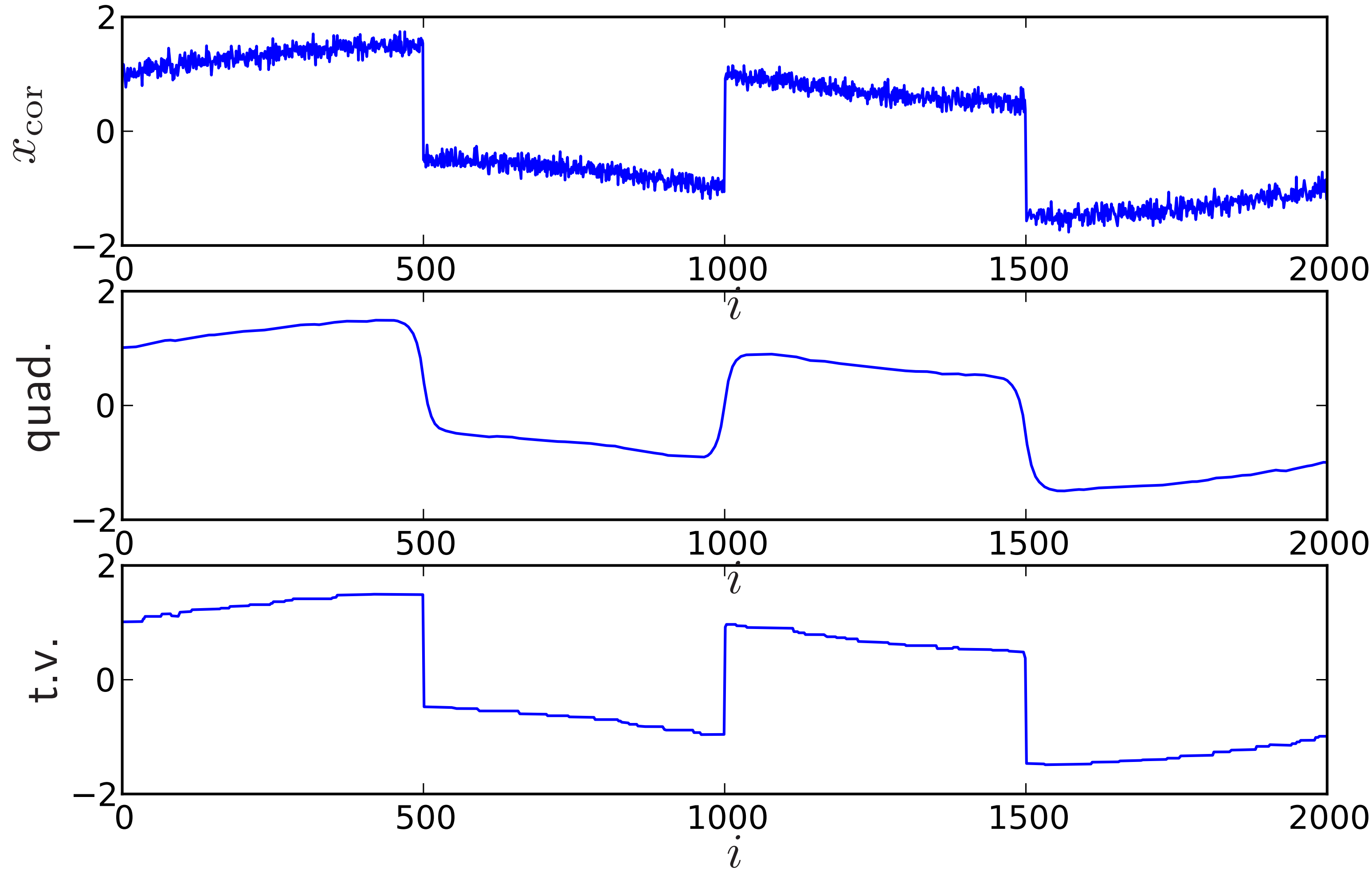
Total variation signal reconstruction

$$\text{minimize} \quad \|\hat{x} - x_{\rm cor}\|_2^2 + \gamma\phi(\hat{x})$$

- $x_{\rm cor} = x + v$ is corrupted version of unknown signal x, with noise v
- variable $\hat{x}$ (reconstructed signal) is estimate of x
- $\phi : \mathbf{R}^n \to \mathbf{R}$ is quadratic or total variation smoothing penalty

$$\phi_{\rm quad}(\hat{x}) = \sum_{i=1}^{n-1}(\hat{x}_{i+1} - \hat{x}_i)^2, \qquad \phi_{\rm tv}(\hat{x}) = \sum_{i=1}^{n-1}|\hat{x}_{i+1} - \hat{x}_i|$$

example: $x_{\rm cor}$, and reconstruction with quadratic and t.v. smoothing



- quadratic smoothing smooths out noise and sharp transitions in signal
- total variation smoothing preserves sharp transitions in signal

Geometric programming

posynomial function

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

with $c_k > 0$

geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \end{array}$$

with f_i posynomial

Geometric program in convex form

change variables to

$$y_i = \log x_i,$$

and take logarithm of cost, constraints

geometric program in convex form:

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \end{array}$$

$b_{ik} = \log c_{ik}$

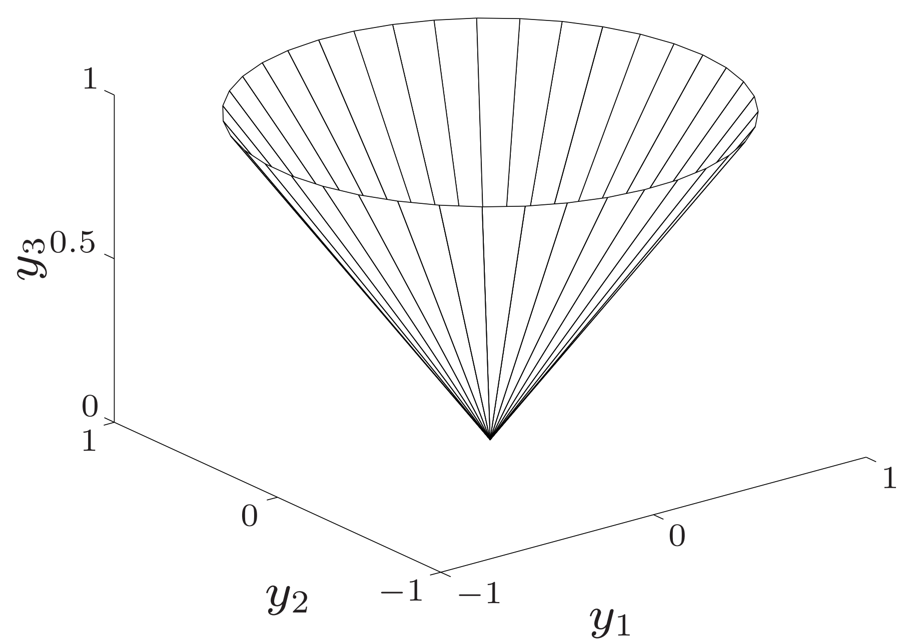
Second-order cone program (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \end{array}$$

- $\|\cdot\|_2$ is Euclidean norm $\|y\|_2 = \sqrt{y_1^2 + \cdots + y_n^2}$
- constraints are nonlinear, nondifferentiable, convex

constraints are inequalities
w.r.t. second-order cone:

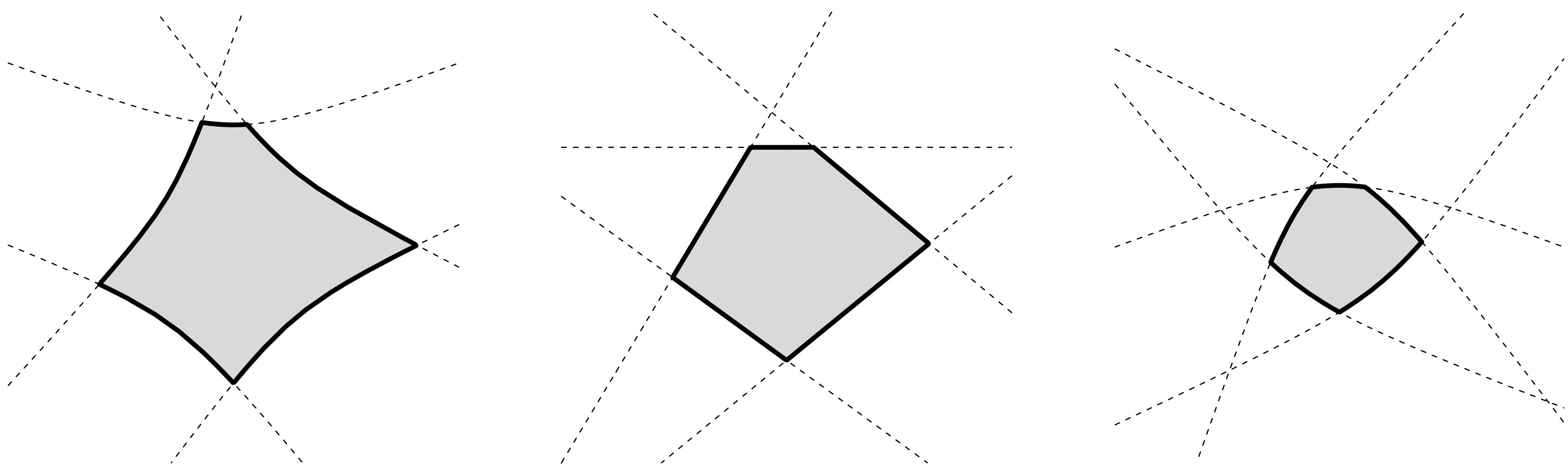
$$\left\{ y \,\middle|\, \sqrt{y_1^2 + \cdots + y_{p-1}^2} \leq y_p \right\}$$



Robust linear program (stochastic)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \dots, m \end{array}$$

- a_i random and normally distributed with mean $\bar{a}_i$, covariance Σ_i
- we require that x satisfies each constraint with probability exceeding η



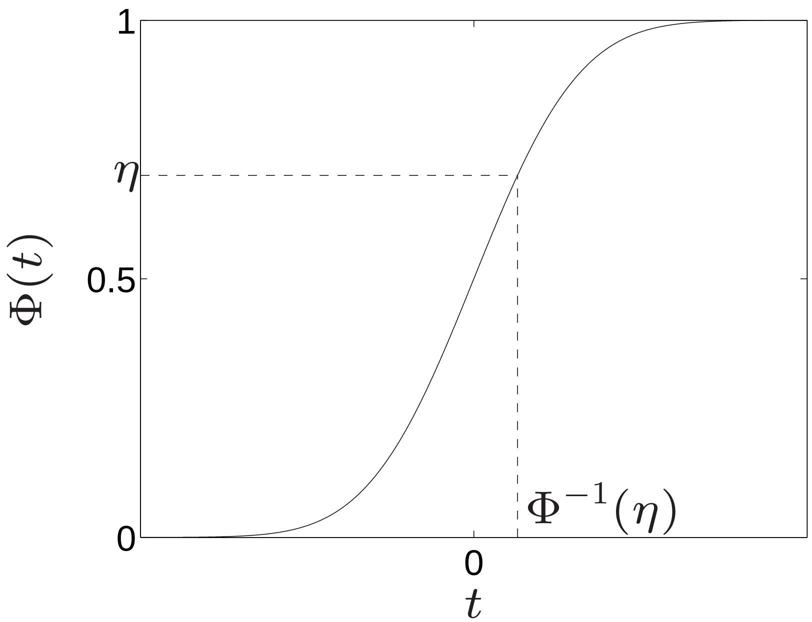
$\eta = 10\%$ $\eta = 50\%$ $\eta = 90\%$

SOCP formulation

the 'chance constraint' $\mathbf{prob}(a_i^T x \leq b_i) \geq \eta$ is equivalent to the constraint

$$\bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \leq b_i$$

Φ is the (unit) normal cumulative density function



robust LP is a second-order cone program for $\eta \geq 0.5$

Robust linear program (deterministic)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

- a_i uncertain but bounded by ellipsoid $\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$
- we require that x satisfies each constraint for all possible a_i

SOCP formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

follows from

$$\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$$

Examples of second-order cone constraints

convex quadratic constraint ($A = LL^T$ positive definite)

$$x^T A x + 2b^T x + c \leq 0$$
$$\Updownarrow$$
$$\left\| L^T x + L^{-1} b \right\|_2 \leq (b^T A^{-1} b - c)^{1/2}$$

extends to positive semidefinite singular A

hyperbolic constraint

$$x^T x \leq yz, \quad y, z \geq 0$$
$$\Updownarrow$$
$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, \quad y, z \geq 0$$

Examples of SOC-representable constraints

positive powers

$$x^{1.5} \le t, \quad x \ge 0$$

$$\Updownarrow$$

$$\exists z: \quad x^2 \le tz, \quad z^2 \le x, \quad x, z \ge 0$$

- two hyperbolic constraints can be converted to SOC constraints
- extends to powers x^p for rational $p \ge 1$

negative powers

$$x^{-3} \le t, \quad x > 0$$

$$\Updownarrow$$

$$\exists z: \quad 1 \le tz, \quad z^2 \le tx, \quad x, z \ge 0$$

- two hyperbolic constraints on r.h.s. can be converted to SOC constraints
- extends to powers x^p for rational $p < 0$

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \preceq B \end{array}$$

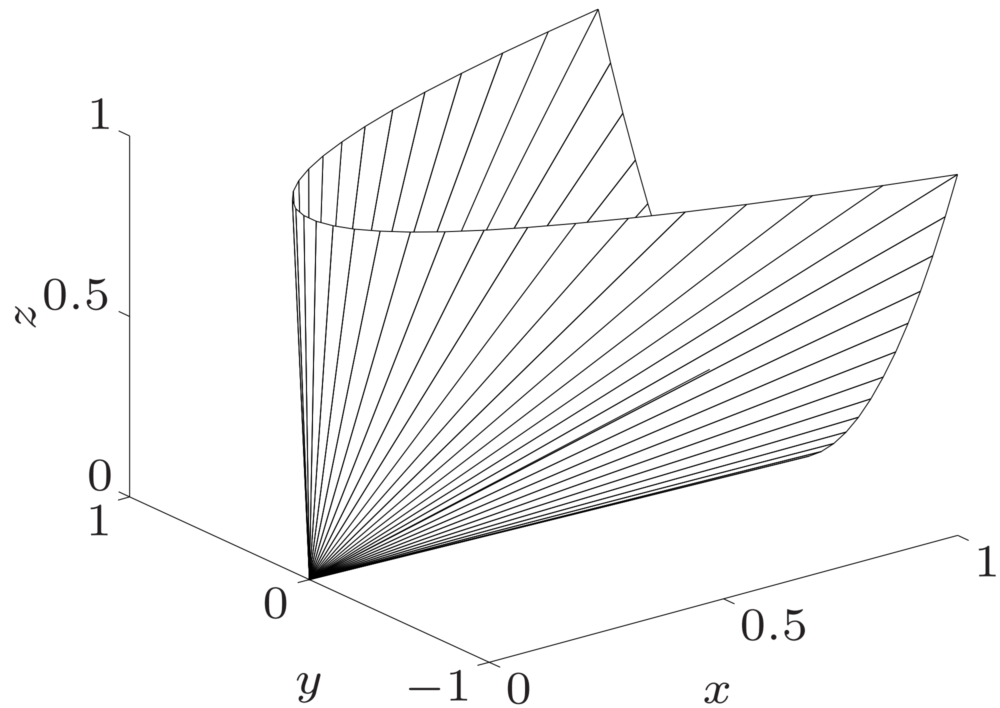
- A_1, A_2, . . . , A_n, B are symmetric matrices
- inequality $X \preceq Y$ means $Y - X$ is *positive semidefinite*, *i.e.*,

$$z^T (Y - X) z = \sum_{i,j} (Y_{ij} - X_{ij}) z_i z_j \geq 0 \text{ for all } z$$

- includes many nonlinear constraints as special cases

Geometry

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0$$



- a nonpolyhedral convex cone
- feasible set of a semidefinite program is the intersection of the positive semidefinite cone in high dimension with planes

Examples

$$A(x) = A_0 + x_1 A_1 + \cdots + x_m A_m \qquad (A_i \in \mathbf{S}^n)$$

eigenvalue minimization (and equivalent SDP)

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(A(x)) \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

matrix-fractional function

$$\begin{array}{ll} \text{minimize} & b^T A(x)^{-1} b \\ \text{subject to} & A(x) \succeq 0 \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} A(x) & b \\ b^T & t \end{array}\right] \succeq 0 \end{array}$$

Matrix norm minimization

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \qquad (A_i \in \mathbf{R}^{p \times q})$$

matrix norm approximation ($\|X\|_2 = \max_k \sigma_k(X)$)

$$\begin{array}{ll} \text{minimize} & \|A(x)\|_2 \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A(x)^T \\ A(x) & tI \end{bmatrix} \succeq 0 \end{array}$$

nuclear norm approximation ($\|X\|_* = \sum_k \sigma_k(X)$)

$$\begin{array}{ll} \text{minimize} & \|A(x)\|_* \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & (\mathbf{tr}\, U + \mathbf{tr}\, V)/2 \\ \text{subject to} & \begin{bmatrix} U & A(x)^T \\ A(x) & V \end{bmatrix} \succeq 0 \end{array}$$

Semidefinite relaxation

semidefinite programming is often used

- to find good bounds for nonconvex polynomial problems, via **relaxation**
- as a heuristic for good suboptimal points

example: Boolean least-squares

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- basic problem in digital communications
- could check all 2^n possible values of $x \in \{-1, 1\}^n$ . . .
- an NP-hard problem, and very hard in general

Lifting

Boolean least-squares problem

$$\begin{array}{ll} \text{minimize} & x^T A^T A x - 2b^T A x + b^T b \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

reformulation: introduce new variable $Y = xx^T$

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A^T A Y) - 2b^T A x + b^T b \\ \text{subject to} & Y = xx^T \\ & \mathbf{diag}(Y) = \mathbf{1} \end{array}$$

- cost function and second constraint are linear (in the variables Y, x)
- first constraint is nonlinear and nonconvex

. . . still a very hard problem

Relaxation

replace $Y = xx^T$ with weaker constraint $Y \succeq xx^T$ to obtain relaxation

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A^TAY) - 2b^TAx + b^Tb \\ \text{subject to} & Y \succeq xx^T \\ & \mathbf{diag}(Y) = \mathbf{1} \end{array}$$

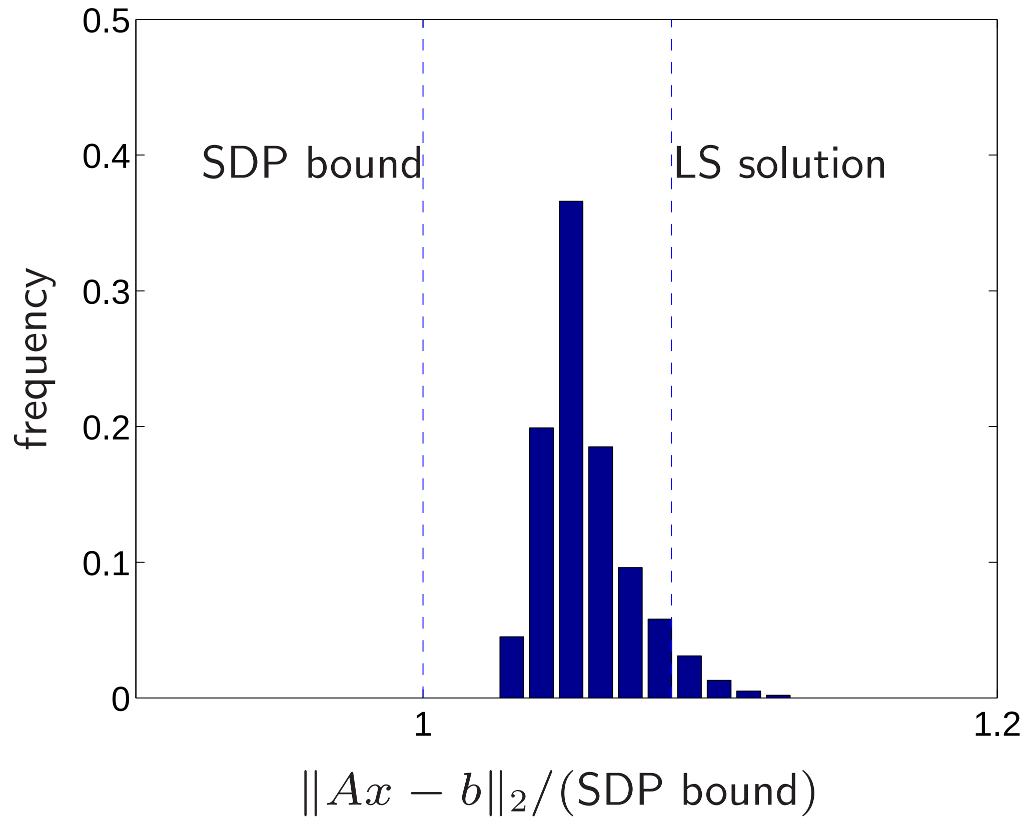
- convex; can be solved as a semidefinite program

$$Y \succeq xx^T \quad \Longleftrightarrow \quad \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

- optimal value gives lower bound for Boolean LS problem
- if $Y = xx^T$ at the optimum, we have solved the exact problem
- otherwise, can use *randomized rounding*

  generate z from $\mathcal{N}(x, Y - xx^T)$ and take $x = \mathbf{sign}(z)$

Example



- $n = 100$: feasible set has $2^{100} \approx 10^{30}$ points
- histogram of 1000 randomized solutions from SDP relaxation

Overview

1. Basic theory and convex modeling

- convex sets and functions
- common problem classes and applications

2. Interior-point methods for conic optimization

- conic optimization
- barrier methods
- symmetric primal-dual methods

3. First-order methods

- (proximal) gradient algorithms
- dual techniques and multiplier methods

Conic optimization

- definitions and examples
- modeling
- duality

Generalized (conic) inequalities

conic inequality: a constraint $x \in K$ with K a convex cone in $\mathbf{R}^m$

we require that K is a **proper** cone:

- closed
- pointed: does not contain a line (equivalently, $K \cap (-K) = \{0\}$
- with nonempty interior: $\mathbf{int}\, K \neq \emptyset$ (equivalently, $K + (-K) = \mathbf{R}^m$)

notation

$$x \succeq_K y \quad \Longleftrightarrow \quad x - y \in K, \qquad\qquad x \succ_K y \quad \Longleftrightarrow \quad x - y \in \mathbf{int}\, K$$

subscript in $\succeq_K$ is omitted if K is clear from the context

Cone linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq_K b \end{array}$$

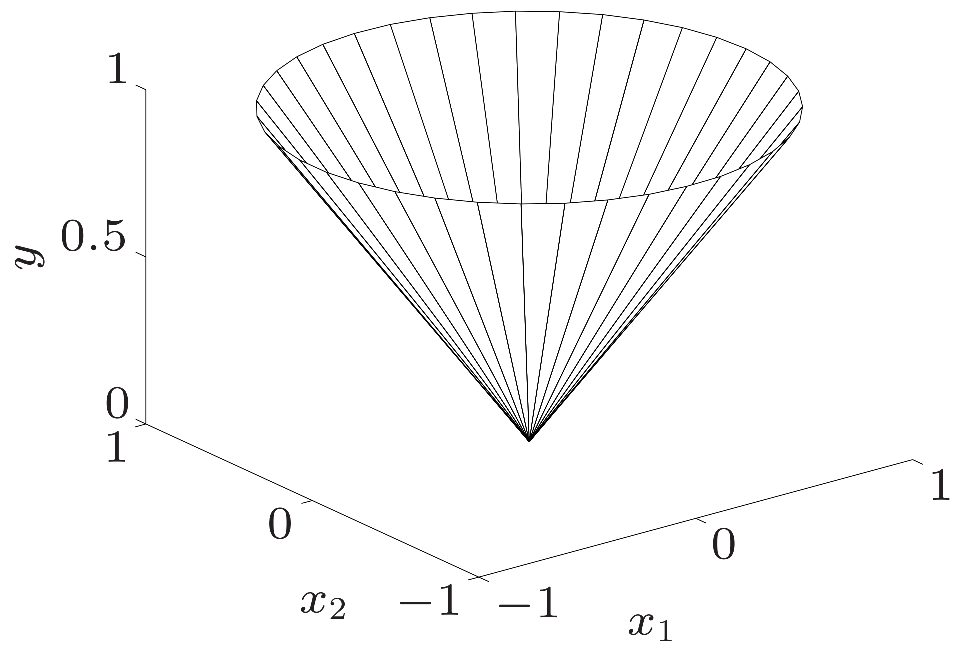
if K is the nonnegative orthant, this is a (regular) linear program

widely used in recent literature on convex optimization

- **modeling:** a small number of 'primitive' cones is sufficient to express most convex constraints that arise in practice
- **algorithms**: a convenient problem format when extending interior-point algorithms for linear programming to convex optimization

Norm cone

$$K = \left\{(x, y) \in \mathbf{R}^{m-1} \times \mathbf{R} \mid \|x\| \leq y\right\}$$



for the Euclidean norm this is the second-order cone (notation: $\mathcal{Q}^m$)

Second-order cone program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|B_{k0}x + d_{k0}\|_2 \leq B_{k1}x + d_{k1}, \quad k = 1, \ldots, r \end{array}$$

cone LP formulation: express constraints as $Ax \preceq_K b$

$$K = \mathcal{Q}^{m_1} \times \cdots \times \mathcal{Q}^{m_r}, \qquad A = \begin{bmatrix} -B_{10} \\ -B_{11} \\ \vdots \\ -B_{r0} \\ -B_{r1} \end{bmatrix}, \qquad b = \begin{bmatrix} d_{10} \\ d_{11} \\ \vdots \\ d_{r0} \\ d_{r1} \end{bmatrix}$$

(assuming B_{k0}, d_{k0} have $m_k - 1$ rows)

Vector notation for symmetric matrices

- vectorized symmetric matrix: for $U \in \mathbf{S}^p$

$$\mathbf{vec}(U) = \sqrt{2}\left(\frac{U_{11}}{\sqrt{2}}, U_{21}, \ldots, U_{p1}, \frac{U_{22}}{\sqrt{2}}, U_{32}, \ldots, U_{p2}, \ldots, \frac{U_{pp}}{\sqrt{2}}\right)$$

- inverse operation: for $u = (u_1, u_2, \ldots, u_n) \in \mathbf{R}^n$ with $n = p(p+1)/2$

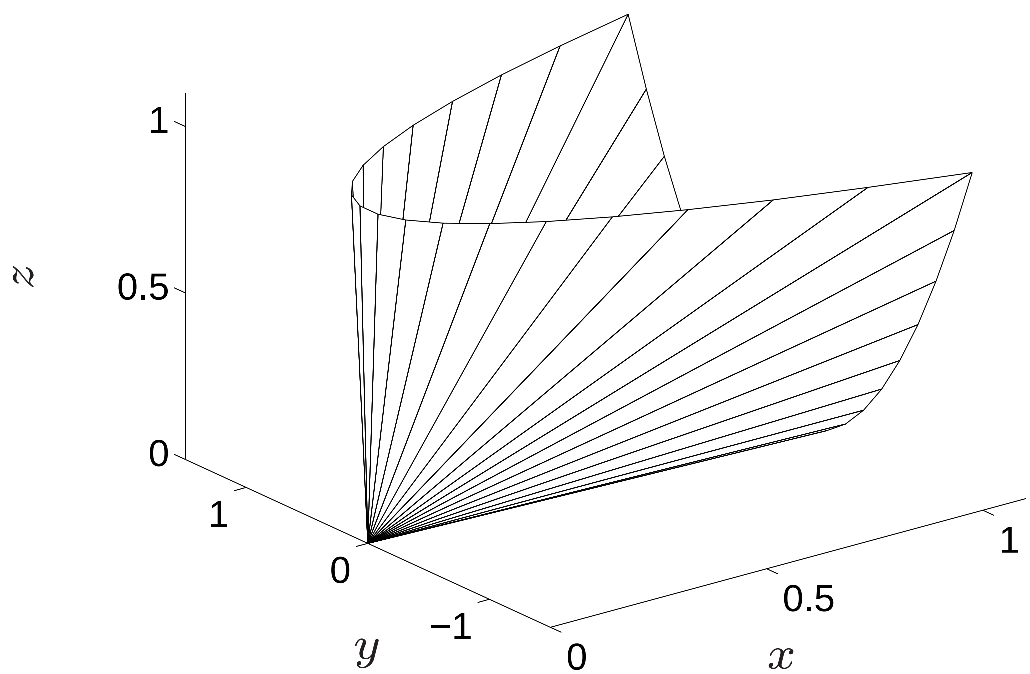
$$\mathbf{mat}(u) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}u_1 & u_2 & \cdots & u_p \\ u_2 & \sqrt{2}u_{p+1} & \cdots & u_{2p-1} \\ \vdots & \vdots & & \vdots \\ u_p & u_{2p-1} & \cdots & \sqrt{2}u_{p(p+1)/2} \end{bmatrix}$$

coefficients $\sqrt{2}$ are added so that standard inner products are preserved:

$$\mathbf{tr}(UV) = \mathbf{vec}(U)^T \mathbf{vec}(V), \qquad u^T v = \mathbf{tr}(\mathbf{mat}(u)\,\mathbf{mat}(v))$$

Positive semidefinite cone

$$\mathcal{S}^p = \{\mathbf{vec}(X) \mid X \in \mathbf{S}^p_+\} = \{x \in \mathbf{R}^{p(p+1)/2} \mid \mathbf{mat}(x) \succeq 0\}$$



$$\mathcal{S}^2 = \left\{(x, y, z) \;\middle|\; \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} \succeq 0\right\}$$

Semidefinite program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 A_{11} + x_2 A_{12} + \cdots + x_n A_{1n} \preceq B_1 \\ & \ldots \\ & x_1 A_{r1} + x_2 A_{r2} + \cdots + x_n A_{rn} \preceq B_r \end{array}$$

r linear matrix inequalities of order $p_1, \ldots, p_r$

cone LP formulation: express constraints as $Ax \preceq_K B$

$$K = \mathcal{S}^{p_1} \times \mathcal{S}^{p_2} \times \cdots \times \mathcal{S}^{p_r}$$

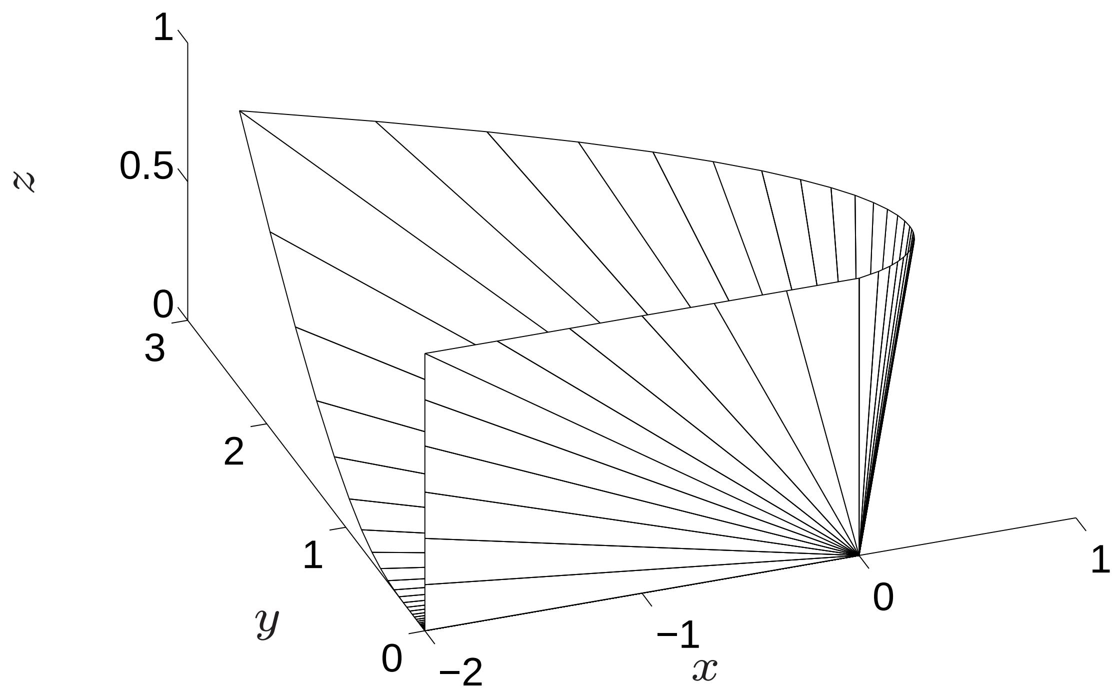
$$A = \left[\begin{array}{cccc} \mathbf{vec}(A_{11}) & \mathbf{vec}(A_{12}) & \cdots & \mathbf{vec}(A_{1n}) \\ \mathbf{vec}(A_{21}) & \mathbf{vec}(A_{22}) & \cdots & \mathbf{vec}(A_{2n}) \\ \vdots & \vdots & & \vdots \\ \mathbf{vec}(A_{r1}) & \mathbf{vec}(A_{r2}) & \cdots & \mathbf{vec}(A_{rn}) \end{array} \right], \qquad b = \left[\begin{array}{c} \mathbf{vec}(B_1) \\ \mathbf{vec}(B_2) \\ \vdots \\ \mathbf{vec}(B_r) \end{array} \right]$$

Exponential cone

the epigraph of the perspective of $\exp x$ is a non-proper cone

$$K = \left\{(x, y, z) \in \mathbf{R}^3 \mid ye^{x/y} \leq z,\ y > 0\right\}$$

the exponential cone is $K_{\exp} = \mathbf{cl}\, K = K \cup \{(x, 0, z) \mid x \leq 0, z \geq 0\}$



Geometric program

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \log \sum_{k=1}^{n_i} \exp(a_{ik}^T x + b_{ik}) \leq 0, \quad i = 1, \ldots, r
\end{array}
$$

cone LP formulation

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \begin{bmatrix} a_{ik}^T x + b_{ik} \\ 1 \\ z_{ik} \end{bmatrix} \in K_{\exp}, \quad k = 1, \ldots, n_i, \quad i = 1, \ldots, r \\
& \sum_{k=1}^{n_i} z_{ik} \leq 1, \quad i = 1, \ldots, m
\end{array}
$$

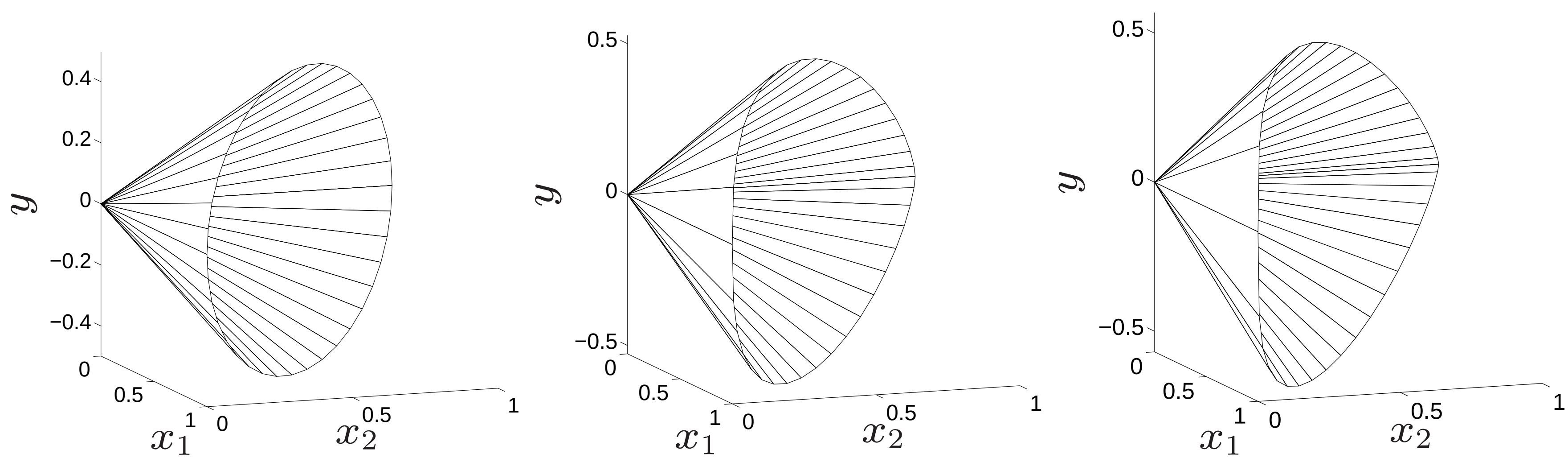
Power cone

definition: for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) > 0$, $\sum_{i=1}^{m} \alpha_i = 1$

$$K_\alpha = \left\{(x, y) \in \mathbf{R}_+^m \times \mathbf{R} \mid |y| \leq x_1^{\alpha_1} \cdots x_m^{\alpha_m}\right\}$$

examples for $m = 2$

$\alpha = (\frac{1}{2}, \frac{1}{2})$ $\qquad$ $\alpha = (\frac{2}{3}, \frac{1}{3})$ $\qquad$ $\alpha = (\frac{3}{4}, \frac{1}{4})$



Outline

- definition and examples
- **modeling**
- duality

Modeling software

modeling packages for convex optimization

- CVX, YALMIP (MATLAB)
- CVXPY, CVXMOD (Python)

assist the user in formulating convex problems, by automating two tasks:

- verifying convexity from convex calculus rules
- transforming problem in input format required by standard solvers

related packages

general-purpose optimization modeling: AMPL, GAMS

CVX example

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_1 \\ \text{subject to} & 0 \le x_k \le 1, \quad k = 1, \ldots, n \end{array}$$

MATLAB code

```
cvx_begin
    variable x(3);
    minimize(norm(A*x - b, 1))
    subject to
         x >= 0;
         x <= 1;
cvx_end
```

- between `cvx_begin` and `cvx_end`, `x` is a CVX variable
- after execution, `x` is MATLAB variable with optimal solution

Modeling and conic optimization

convex modeling systems (CVX, YALMIP, CVXPY, CVXMOD, . . .)

- convert problems stated in standard mathematical notation to cone LPs
- in principle, any convex problem can be represented as a cone LP
- in practice, a small set of primitive cones is used ($\mathbf{R}^n_+$, $\mathcal{Q}^p$, $\mathcal{S}^p$)
- choice of cones is limited by available algorithms and solvers (see later)

modeling systems implement set of rules for expressing constraints

$$f(x) \leq t$$

as conic inequalities for the implemented cones

Examples of second-order cone representable functions

- convex quadratic

$$f(x) = x^T P x + q^T x + r \qquad (P \succeq 0)$$

- quadratic-over-linear function

$$f(x, y) = \frac{x^T x}{y} \quad \text{with } \mathbf{dom}\, f = \mathbf{R}^n \times \mathbf{R}_+ \qquad (\text{assume } 0/0 = 0)$$

- convex powers with rational exponent

$$f(x) = |x|^\alpha, \qquad f(x) = \begin{cases} x^\beta & x > 0 \\ +\infty & x \leq 0 \end{cases}$$

  for rational $\alpha \geq 1$ and $\beta \leq 0$

- p-norm $f(x) = \|x\|_p$ for rational $p \geq 1$

Examples of SD cone representable functions

- matrix-fractional function

$$f(X,y) = y^T X^{-1} y \quad \text{with } \mathbf{dom}\, f = \{(X,y) \in \mathbf{S}^n_+ \times \mathbf{R}^n \mid y \in \mathcal{R}(X)\}$$

- maximum eigenvalue of symmetric matrix

- maximum singular value $f(X) = \|X\|_2 = \sigma_1(X)$

$$\|X\|_2 \le t \quad \Longleftrightarrow \quad \left[\begin{array}{cc} tI & X \\ X^T & tI \end{array}\right] \succeq 0$$

- nuclear norm $f(X) = \|X\|_* = \sum_i \sigma_i(X)$

$$\|X\|_* \le t \quad \Longleftrightarrow \quad \exists U,V : \left[\begin{array}{cc} U & X \\ X^T & V \end{array}\right] \succeq 0, \quad \frac{1}{2}(\mathbf{tr}\, U + \mathbf{tr}\, V) \le t$$

Functions representable with exponential and power cone

exponential cone

- exponential and logarithm
- entropy $f(x) = x \log x$

power cone

- increasing power of absolute value: $f(x) = |x|^p$ with $p \geq 1$
- decreasing power: $f(x) = x^q$ with $q \leq 0$ and domain $\mathbf{R}_{++}$
- p-norm: $f(x) = \|x\|_p$ with $p \geq 1$

Outline

- definition and examples
- modeling
- **duality**

Linear programming duality

primal and dual LP

$$
\begin{array}{lll}
\text{(P)} & \text{minimize} & c^T x \\
& \text{subject to} & Ax \leq b
\end{array}
\qquad\qquad
\begin{array}{lll}
\text{(D)} & \text{maximize} & -b^T z \\
& \text{subject to} & A^T z + c = 0 \\
& & z \geq 0
\end{array}
$$

- primal optimal value is $p^\star$ ($+\infty$ if infeasible, $-\infty$ if unbounded below)
- dual optimal value is $d^\star$ ($-\infty$ if infeasible, $+\infty$ if unbounded below)

duality theorem

- weak duality: $p^\star \geq d^\star$, with no exception
- strong duality: $p^\star = d^\star$ if primal or dual is feasible
- if $p^\star = d^\star$ is finite, then primal and dual optima are attained

Dual cone

definition

$$K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}$$

K^* is a proper cone if K is a proper cone

dual inequality: $x \succeq_* y$ means $x \succeq_{K^*} y$ for generic proper cone K

note: dual cone depends on choice of inner product:

$$H^{-1}K^*$$

is dual cone for inner product $\langle x, y \rangle = x^T H y$

Examples

- $\mathbf{R}^p_+$, $\mathcal{Q}^p$, $\mathcal{S}^p$ are self-dual: $K = K^*$

- dual of a norm cone is the norm cone of the dual norm

- dual of exponential cone

$$K^*_{\exp} = \left\{(u, v, w) \in \mathbf{R}_- \times \mathbf{R} \times \mathbf{R}^+ \mid -u\log(-u/w) + u - v \leq 0\right\}$$

(with $0\log(0/w) = 0$ if $w \geq 0$)

- dual of power cone is

$$K^*_\alpha = \left\{(u, v) \in \mathbf{R}^m_+ \times \mathbf{R} \mid |v| \leq (u_1/\alpha_1)^{\alpha_1} \cdots (u_m/\alpha_m)^{\alpha_m}\right\}$$

Primal and dual cone LP

primal problem (optimal value $p^\star$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

dual problem (optimal value $d^\star$)

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \succeq_* 0 \end{array}$$

weak duality: $p^\star \geq d^\star$ (without exception)

Strong duality

$$p^\star = d^\star$$

if primal or dual is strictly feasible

- slightly weaker than LP duality (which only requires feasibility)
- can have $d^\star < p^\star$ with finite $p^\star$ and $d^\star$

other implications of strict feasibility

- if primal is strictly feasible, then dual optimum is attained (if $d^\star$ is finite)
- if dual is strictly feasible, then primal optimum is attained (if $p^\star$ is finite)

Optimality conditions

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax + s = b \\
& s \succeq 0
\end{array}
\qquad\qquad
\begin{array}{ll}
\text{maximize} & -b^T z \\
\text{subject to} & A^T z + c = 0 \\
& z \succeq_* 0
\end{array}
$$

optimality conditions

$$
\left[\begin{array}{c} 0 \\ s \end{array}\right] = \left[\begin{array}{cc} 0 & A^T \\ -A & 0 \end{array}\right] \left[\begin{array}{c} x \\ z \end{array}\right] + \left[\begin{array}{c} c \\ b \end{array}\right]
$$

$$
s \succeq 0, \qquad z \succeq_* 0, \qquad z^T s = 0
$$

duality gap: inner product of (x, z) and $(0, s)$ gives

$$
z^T s = c^T x + b^T z
$$

Barrier methods

- barrier method for linear programming
- normal barriers
- barrier method for conic optimization

History

- 1960s: Sequentially Unconstrained Minimization Technique (SUMT)

  solves nonlinear convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

  via a sequence of unconstrained minimization problems

$$\text{minimize} \quad t f_0(x) - \sum_{i=1}^{m} \log(-f_i(x))$$

- 1980s: LP barrier methods with polynomial worst-case complexity

- 1990s: barrier methods for non-polyhedral cone LPs

Logarithmic barrier function for linear inequalities

- barrier for nonnegative orthant $\mathbf{R}_+^m$: $\phi(s) = -\sum_{i=1}^{m} \log s_i$
- barrier for inequalities $Ax \leq b$:

$$\psi(x) = \phi(b - Ax) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

convex, $\psi(x) \to \infty$ at boundary of $\mathbf{dom}\,\psi = \{x \mid Ax < b\}$

gradient and Hessian

$$\nabla\psi(x) = -A^T \nabla\phi(s), \qquad \nabla^2\psi(x) = A^T \nabla\phi^2(s) A$$

with $s = b - Ax$ and

$$\nabla\phi(s) = -\left(\frac{1}{s_1}, \ldots, \frac{1}{s_m}\right), \qquad \nabla\phi^2(s) = \mathbf{diag}\left(\frac{1}{s_1^2}, \ldots, \frac{1}{s_m^2}\right)$$

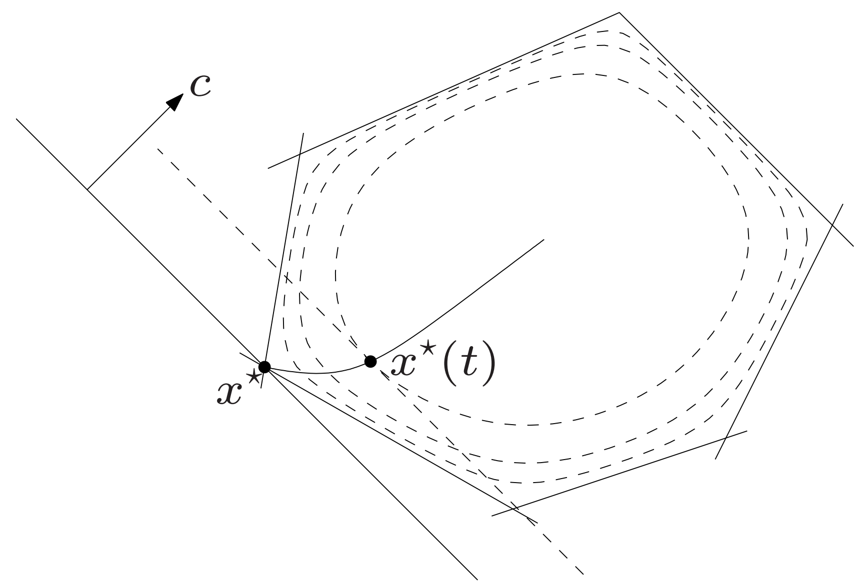
Central path for linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

central path: minimizers $x^\star(t)$ of

$$f_t(x) = tc^T x + \phi(b - Ax)$$

t is a positive parameter



optimality conditions: $x = x^\star(t)$ satisfies

$$\nabla f_t(x) = tc - A^T \nabla\phi(s) = 0, \qquad s = b - Ax$$

Central path and duality

dual feasible point on central path

- for $x = x^\star(t)$ and $s = b - Ax$,

$$z^*(t) = -\frac{1}{t}\nabla\phi(s) = \left(\frac{1}{ts_1}, \frac{1}{ts_2}, \ldots, \frac{1}{ts_m}\right)$$

  $z = z^\star(t)$ is strictly dual feasible: $c + A^T z = 0$ and $z > 0$

- can be corrected to account for inexact centering of $x \approx x^\star(t)$

duality gap between $x = x^\star(t)$ and $z = z^\star(t)$ is

$$c^T x + b^T z = s^T z = \frac{m}{t}$$

gives bound on suboptimality: $c^T x^\star(t) - p^\star \leq m/t$

Barrier method

starting with $t > 0$, strictly feasible x

- make one or more Newton steps to (approximately) minimize f_t:

$$x^+ = x - \alpha \nabla^2 f_t(x)^{-1} \nabla f_t(x)$$

  step size α is fixed or from line search

- increase t and repeat until $c^T x - p^\star \leq \epsilon$

complexity: with proper initialization, step size, update scheme for t,

$$\#\text{Newton steps} = O\left(\sqrt{m} \log(1/\epsilon)\right)$$

result follows from convergence analysis of Newton's method for f_t

Outline

- barrier method for linear programming
- **normal barriers**
- barrier method for conic optimization

Normal barrier for proper cone

ϕ is a θ-normal barrier for the proper cone K if it is

- a **barrier**: smooth, convex, domain $\mathbf{int}\, K$, blows up at boundary of K
- **logarithmically homogeneous** with parameter θ:

$$\phi(tx) = \phi(x) - \theta \log t, \qquad \forall x \in \mathbf{int}\, K,\ t > 0$$

- **self-concordant**: restriction $g(\alpha) = \phi(x + \alpha v)$ to any line satisfies

$$g'''(\alpha) \leq 2g''(\alpha)^{3/2}$$

(Nesterov and Nemirovski, 1994)

Examples

nonnegative orthant: $K = \mathbf{R}^m_+$

$$\phi(x) = -\sum_{i=1}^{m} \log x_i \qquad (\theta = m)$$

second-order cone: $K = \mathcal{Q}^p = \{(x, y) \in \mathbf{R}^{p-1} \times \mathbf{R} \mid \|x\|_2 \leq y\}$

$$\phi(x, y) = -\log(y^2 - x^T x) \qquad (\theta = 2)$$

semidefinite cone: $K = \mathcal{S}^m = \{x \in \mathbf{R}^{m(m+1)/2} \mid \mathbf{mat}(x) \succeq 0\}$

$$\phi(x) = -\log\det \mathbf{mat}(x) \qquad (\theta = m)$$

exponential cone: $K_{\exp} = \mathbf{cl}\{(x,y,z) \in \mathbf{R}^3 \mid ye^{x/y} \leq z,\ y > 0\}$

$$\phi(x,y,z) = -\log\left(y\log(z/y) - x\right) - \log z - \log y \qquad (\theta = 3)$$

power cone: $K = \{(x_1, x_2, y) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R} \mid |y| \leq x_1^{\alpha_1} x_2^{\alpha_2}\}$

$$\phi(x,y) = -\log\left(x_1^{2\alpha_1} x_2^{2\alpha_2} - y^2\right) - \log x_1 - \log x_2 \qquad (\theta = 4)$$

Central path

conic LP (with inequality with respect to proper cone K)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

barrier for the feasible set

$$\phi(b - Ax)$$

where ϕ is a θ-normal barrier for K

central path: set of minimizers $x^\star(t)$ (with $t > 0$) of

$$f_t(x) = tc^T x + \phi(b - Ax)$$

Newton step

centering problem

$$\text{minimize} \quad f_t(x) = tc^T x + \phi(b - Ax)$$

Newton step at x

$$\Delta x = -\nabla^2 f_t(x)^{-1} \nabla f_t(x)$$

Newton decrement

$$\begin{aligned} \lambda_t(x) &= \left(\Delta x^T \nabla^2 f_t(x) \Delta x\right)^{1/2} \\ &= \left(-\nabla f_t(x)^T \Delta x\right)^{1/2} \end{aligned}$$

useful as a measure of proximity of x to $x^\star(t)$

Damped Newton method

$$\text{minimize} \quad f_t(x) = tc^T x + \phi(b - Ax)$$

algorithm (with parameters $\epsilon \in (0, 1/2)$, $\eta \in (0, 1/4]$)

select a starting point $x \in \mathbf{dom}\, f_t$

repeat:

1. compute Newton step Δx and Newton decrement $\lambda_t(x)$
2. if $\lambda_t(x)^2 \le \epsilon$, return x
3. otherwise, set $x := x + \alpha \Delta x$ with

$$\alpha = \frac{1}{1 + \lambda_t(x)} \quad \text{if } \lambda_t(x) \ge \eta, \qquad \alpha = 1 \quad \text{if } \lambda_t(x) < \eta$$

- stopping criterion $\lambda_t(x)^2 \le \epsilon$ implies $f_t(x) - \inf f_t(x) \le \epsilon$
- alternatively, can use backtracking line search

Convergence results for damped Newton method

- **damped Newton phase:** f_t decreases by at least a positive constant γ

$$f_t(x^+) - f_t(x) \leq -\gamma \quad \text{if } \lambda_t(x) \geq \eta$$

where $\gamma = \eta - \log(1 + \eta)$

- **quadratic convergence phase:** λ_t rapidly decreases to zero

$$2\lambda_t(x^+) \leq (2\lambda_t(x))^2 \quad \text{if } \lambda_t(x) < \eta$$

implies $\lambda_t(x^+) \leq 2\eta^2 < \eta$

conclusion: the number of Newton iterations is bounded by

$$\frac{f_t(x^{(0)}) - \inf f_t(x)}{\gamma} + \log_2 \log_2(1/\epsilon)$$

Outline

- barrier method for linear programming
- normal barriers
- **barrier method for conic optimization**

Central path and duality

$$x^\star(t) = \operatorname{argmin}\left(tc^T x + \phi(b - Ax)\right)$$

duality point on central path: $x^\star(t)$ defines a strictly dual feasible $z^\star(t)$

$$z^\star(t) = -\frac{1}{t}\nabla\phi(s), \qquad s = b - Ax^\star(t)$$

duality gap: gap between $x = x^\star(t)$ and $z = z^\star(t)$ is

$$c^T x + b^T z = s^T z = \frac{\theta}{t}, \qquad c^T x - p^\star \le \frac{\theta}{t}$$

extension near central path (for $\lambda_t(x) < 1$): $c^T x - p^\star \le \left(1 + \dfrac{\lambda_t(x)}{\sqrt{\theta}}\right)\dfrac{\theta}{t}$

(results follow from properties of normal barriers)

Short-step barrier method

algorithm (parameters $\epsilon \in (0,1)$, $\beta = 1/8$)

- select initial x and t with $\lambda_t(x) \leq \beta$
- repeat until $2\theta/t \leq \epsilon$:

$$t := \left(1 + \frac{1}{1+8\sqrt{\theta}}\right) t, \qquad x := x - \nabla f_t(x)^{-1} \nabla f_t(x)$$

properties

- increases t slowly so x stays in region of quadratic region ($\lambda_t(x) \leq \beta$)
- iteration complexity

$$\#\text{iterations} = O\left(\sqrt{\theta} \log\left(\frac{\theta}{\epsilon t_0}\right)\right)$$

- best known worst-case complexity; same as for linear programming

Predictor-corrector methods

short-step barrier methods

- stay in narrow neighborhood of central path (defined by limit on λ_t)
- make small, fixed increases $t^+ = \mu t$

as a result, quite slow in practice

predictor-corrector method

- select new t using a linear approximation to central path ('predictor')
- re-center with new t ('corrector')

allows faster and 'adaptive' increases in t; similar worst-case complexity

Primal-dual methods

- primal-dual algorithms for linear programming
- symmetric cones
- primal-dual algorithms for conic optimization
- implementation

Primal-dual interior-point methods

similarities with barrier method

- follow the same central path
- same linear algebra cost per iteration

differences

- more robust and faster (typically less than 50 iterations)
- primal and dual iterates updated at each iteration
- symmetric treatment of primal and dual iterates
- can start at infeasible points
- include heuristics for adaptive choice of central path parameter t
- often have superlinear asymptotic convergence

Primal-dual central path for linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + s = b \\ & s \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

optimality conditions ($s \circ z$ is component-wise vector product)

$$Ax + s = b, \qquad A^T z + c = 0, \qquad (s, z) \geq 0, \qquad s \circ z = 0$$

primal-dual parametrization of central path

$$Ax + s = b, \qquad A^T z + c = 0, \qquad (s, z) \geq 0, \qquad s \circ z = \mu \mathbf{1}$$

- solution is $x = x^*(t)$, $z = z^*(t)$ for $t = 1/\mu$
- $\mu = (s^T z)/m$ for x, z on the central path

Primal-dual search direction

current iterates $\hat{x}$, $\hat{s} > 0$, $\hat{z} > 0$ updated as

$$\hat{x} := \hat{x} + \alpha\Delta x, \qquad \hat{s} := \hat{s} + \alpha\Delta s, \qquad \hat{z} := \hat{z} + \alpha\Delta z$$

primal and dual steps Δx, Δs, Δz are defined by

$$A(\hat{x} + \Delta x) + \hat{s} + \Delta s = b, \qquad A^T(\hat{z} + \Delta z) + c = 0$$
$$\hat{z} \circ \Delta s + \hat{s} \circ \Delta z = \sigma\hat{\mu}\mathbf{1} - \hat{s} \circ \hat{z}$$

where $\hat{\mu} = (\hat{s}^T\hat{z})/m$ and $\sigma \in [0, 1]$

- last equation is linearization of $(\hat{s} + \Delta s) \circ (\hat{z} + \Delta z) = \sigma\hat{\mu}\mathbf{1}$
- targets point on central path with $\mu = \sigma\hat{\mu}$ *i.e.*, with gap $\sigma(\hat{s}^T\hat{z})$
- different methods use different strategies for selecting σ
- $\alpha \in (0, 1]$ selected so that $\hat{s} > 0$, $\hat{z} > 0$

Linear algebra complexity

at each iteration solve an equation

$$\begin{bmatrix} A & I & 0 \\ 0 & 0 & A^T \\ 0 & \mathbf{diag}(\hat{z}) & \mathbf{diag}(\hat{s}) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta s \\ \Delta z \end{bmatrix} = \begin{bmatrix} b - A\hat{x} - \hat{s} \\ -c - A^T\hat{z} \\ \sigma\hat{\mu}\mathbf{1} - \hat{s} \circ \hat{z} \end{bmatrix}$$

- after eliminating Δs, Δz this reduces to an equation

$$A^T D A \, \Delta x = r,$$

  with $D = \mathbf{diag}(\hat{z}_1/\hat{s}_1, \ldots, \hat{z}_m/\hat{s}_m)$

- similar equation as in simple barrier method (with different D, r)

Outline

- primal-dual algorithms for linear programming
- **symmetric cones**
- primal-dual algorithms for conic optimization
- implementation

Symmetric cones

symmetric primal-dual solvers for cone LPs are limited to **symmetric** cones

- second-order cone
- positive semidefinite cone
- direct products of these 'primitive' symmetric cones (such as $\mathbf{R}^p_+$)

definition: cone of squares $x = y^2 = y \circ y$ for a product $\circ$ that satisfies

1. bilinearity ($x \circ y$ is linear in x for fixed y and vice-versa)
2. $x \circ y = y \circ x$
3. $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$
4. $x^T (y \circ z) = (x \circ y)^T z$

not necessarily associative

Vector product and identity element

nonnegative orthant: component-wise product

$$x \circ y = \mathbf{diag}(x)y$$

identity element is $\mathbf{e} = \mathbf{1} = (1, 1, \ldots, 1)$

positive semidefinite cone: symmetrized matrix product

$$x \circ y = \frac{1}{2}\mathbf{vec}(XY + YX) \quad \text{with } X = \mathbf{mat}(x),\ Y = \mathbf{mat}(Y)$$

identity element is $\mathbf{e} = \mathbf{vec}(I)$

second-order cone: the product of $x = (x_0, x_1)$ and $y = (y_0, y_1)$ is

$$x \circ y = \frac{1}{\sqrt{2}} \begin{bmatrix} x^T y \\ x_0 y_1 + y_0 x_1 \end{bmatrix}$$

identity element is $\mathbf{e} = (\sqrt{2}, 0, \ldots, 0)$

Classification

- symmetric cones are studied in the theory of Euclidean Jordan algebras
- all possible symmetric cones have been characterized

list of symmetric cones

- the second-order cone
- the positive semidefinite cone of Hermitian matrices with real, complex, or quaternion entries
- 3×3 positive semidefinite matrices with octonion entries
- Cartesian products of these 'primitive' symmetric cones (such as $\mathbf{R}^p_+$)

practical implication

can focus on $\mathcal{Q}^p$, $\mathcal{S}^p$ and study these cones using elementary linear algebra

Spectral decomposition

with each symmetric cone/product we associate a 'spectral' decomposition

$$x = \sum_{i=1}^{\theta} \lambda_i q_i, \qquad \text{with} \quad \sum_{i=1}^{\theta} q_i = \mathbf{e} \quad \text{and} \quad q_i \circ q_j = \begin{cases} q_i & i = j \\ 0 & i \neq j \end{cases}$$

semidefinite cone $(K = \mathcal{S}^p)$: eigenvalue decomposition of $\mathbf{mat}(x)$

$$\theta = p, \qquad \mathbf{mat}(x) = \sum_{i=1}^{p} \lambda_i v_i v_i^T, \qquad q_i = \mathbf{vec}(v_i v_i^T)$$

second-order cone $(K = \mathcal{Q}^p)$

$$\theta = 2, \qquad \lambda_i = \frac{x_0 \pm \|x_1\|_2}{\sqrt{2}}, \qquad q_i = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm x_1 / \|x_1\|_2 \end{bmatrix}, \qquad i = 1, 2$$

Applications

nonnegativity

$$x \succeq 0 \iff \lambda_1, \ldots, \lambda_\theta \geq 0, \qquad x \succ 0 \iff \lambda_1, \ldots, \lambda_\theta > 0$$

powers (in particular, inverse and square root)

$$x^\alpha = \sum_i \lambda_i^\alpha q_i$$

log-det barrier

$$\phi(x) = -\log\det x = -\sum_{i=1}^{\theta} \log \lambda_i$$

a θ-normal barrier, with gradient $\nabla\phi(x) = -x^{-1}$

Outline

- primal-dual algorithms for linear programming
- symmetric cones
- **primal-dual algorithms for conic optimization**
- implementation

Symmetric parametrization of central path

centering problem

$$\text{minimize} \quad tc^T x + \phi(b - Ax)$$

optimality conditions (using $\nabla\phi(s) = -s^{-1}$)

$$Ax + s = b, \qquad A^T z + c = 0, \qquad (s, z) \succ 0, \qquad z = \frac{1}{t} s^{-1}$$

equivalent symmetric form (with $\mu = 1/t$)

$$Ax + b = s, \qquad A^T z + c = 0, \qquad (s, z) \succ 0, \qquad s \circ z = \mu\, \mathbf{e}$$

Scaling with Hessian

linear transformation with $H = \nabla^2\phi(u)$ has several important properties

- preserves conic inequalities: $s \succ 0 \Longleftrightarrow Hs \succ 0$
- if s is invertible, then Hs is invertible and $(Hs)^{-1} = H^{-1}s^{-1}$
- preserves central path:

$$s \circ z = \mu\, \mathbf{e} \quad \Longleftrightarrow \quad (Hs) \circ (H^{-1}z) = \mu\, \mathbf{e}$$

example ($K = \mathcal{S}^p$): transformation $w = \nabla^2\phi(u)s$ is a congruence

$$W = U^{-1}SU^{-1}, \qquad W = \mathbf{mat}(w), \quad S = \mathbf{mat}(s), \quad U = \mathbf{mat}(u)$$

Primal-dual search direction

steps Δx, Δs, Δz at current iterates $\hat{x}$, $\hat{s}$, $\hat{z}$ are defined by

$$A(\hat{x} + \Delta x) + \hat{s} + \Delta s = b, \qquad A^T(\hat{z} + \Delta z) + c = 0$$

$$(H\hat{s}) \circ (H^{-1}\Delta z) + (H^{-1}\hat{z}) \circ (H\Delta s) = \sigma\hat{\mu}\mathbf{e} - (H\hat{s}) \circ (H^{-1}\hat{z})$$

where $\hat{\mu} = (\hat{s}^T\hat{z})/\theta$, $\sigma \in [0, 1]$, and $H = \nabla^2\phi(u)$

- last equation is linearization of

$$(H(\hat{s} + \Delta s)) \circ \left(H^{-1}(\hat{z} + \Delta z)\right) = \sigma\hat{\mu}\mathbf{e}$$

- different algorithms use different choices of σ, H
- Nesterov-Todd scaling: choose $H = \nabla^2\phi(u)$ such that $H\hat{s} = H^{-1}\hat{z}$

Outline

- primal-dual algorithms for linear programming
- symmetric cones
- primal-dual algorithms for conic optimization
- **implementation**

Software implementations

general-purpose software for nonlinear convex optimization

- several high-quality packages (MOSEK, Sedumi, SDPT3, SDPA, . . .)
- exploit sparsity to achieve scalability

customized implementations

- can exploit non-sparse types of problem structure
- often orders of magnitude faster than general-purpose solvers

Example: ℓ_1-regularized least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2 + \|x\|_1$$

A is $m \times n$ (with $m \leq n$) and dense

quadratic program formulation

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 + \mathbf{1}^T u \\ \text{subject to} & -u \leq x \leq u \end{array}$$

- coefficient of Newton system in interior-point method is

$$\left[\begin{array}{cc} A^T A & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} D_1 + D_2 & D_2 - D_1 \\ D_2 - D_1 & D_1 + D_2 \end{array}\right] \qquad (D_1,\ D_2 \text{ positive diagonal})$$

- expensive for large n: cost is $O(n^3)$

customized implementation

- can reduce Newton equation to solution of a system

$$(AD^{-1}A^T + I)\Delta u = r$$

- cost per iteration is $O(m^2 n)$

comparison (seconds on 2.83 Ghz Core 2 Quad machine)

m	n	custom	general-purpose
50	200	0.02	0.32
50	400	0.03	0.59
100	1000	0.12	1.69
100	2000	0.24	3.43
500	1000	1.19	7.54
500	2000	2.38	17.6

custom solver is CVXOPT; general-purpose solver is MOSEK

Overview

1. Basic theory and convex modeling

- convex sets and functions
- common problem classes and applications

2. Interior-point methods for conic optimization

- conic optimization
- barrier methods
- symmetric primal-dual methods

3. First-order methods

- (proximal) gradient algorithms
- dual techniques and multiplier methods

Gradient methods

- gradient and subgradient method
- proximal gradient method
- fast proximal gradient methods

Classical gradient method

to minimize a convex differentiable function f: choose $x^{(0)}$ and repeat

$$x^{(k)} = x^{(k-1)} - t_k \nabla f(x^{(k-1)}), \qquad k = 1, 2, \ldots$$

step size t_k is constant or from line search

advantages

- every iteration is inexpensive
- does not require second derivatives

disadvantages

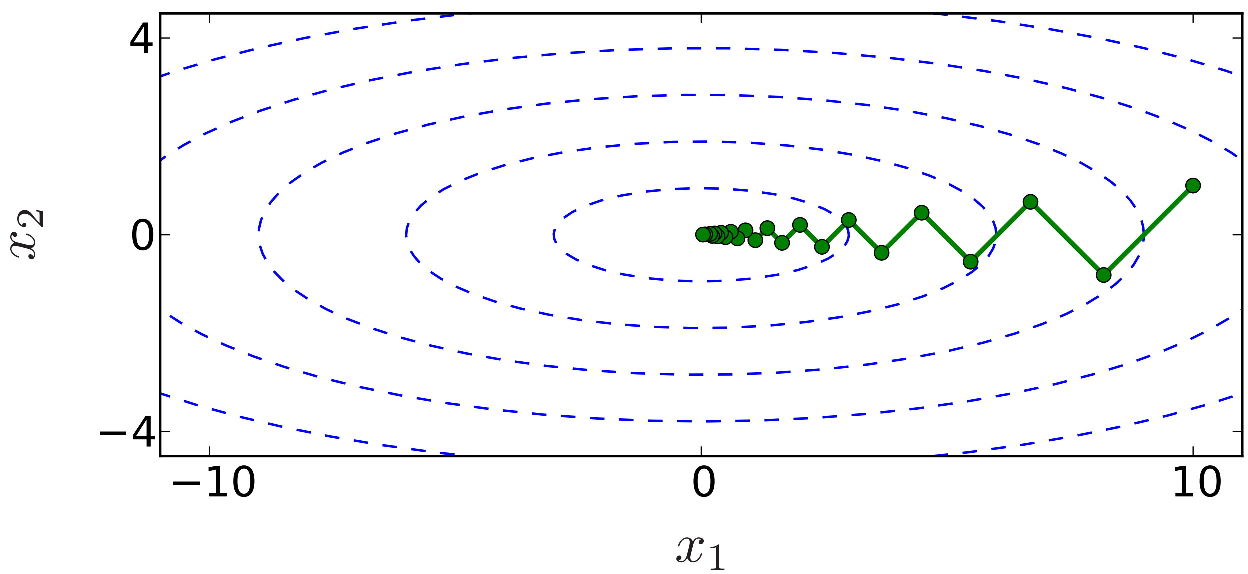
- often very slow; very sensitive to scaling
- does not handle nondifferentiable functions

Quadratic example

$$f(x) = \frac{1}{2}(x_1^2 + \gamma x_2^2) \qquad (\gamma > 1)$$

with exact line search and starting point $x^{(0)} = (\gamma, 1)$

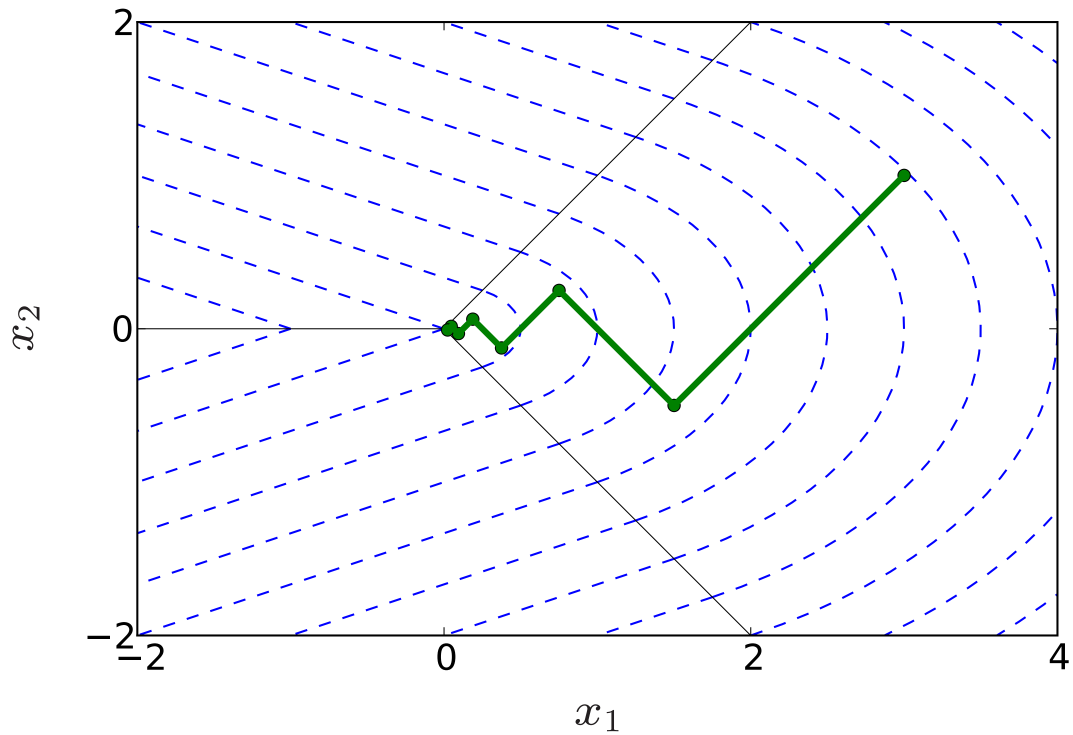
$$\frac{\|x^{(k)} - x^\star\|_2}{\|x^{(0)} - x^\star\|_2} = \left(\frac{\gamma - 1}{\gamma + 1}\right)^k$$



Nondifferentiable example

$$f(x) = \sqrt{x_1^2 + \gamma x_2^2} \quad (|x_2| \le x_1), \qquad f(x) = \frac{x_1 + \gamma |x_2|}{\sqrt{1+\gamma}} \quad (|x_2| > x_1)$$

with exact line search, $x^{(0)} = (\gamma, 1)$, converges to non-optimal point



First-order methods

address one or both disadvantages of the gradient method

methods for nondifferentiable or constrained problems

- smoothing methods
- subgradient method
- proximal gradient method

methods with improved convergence

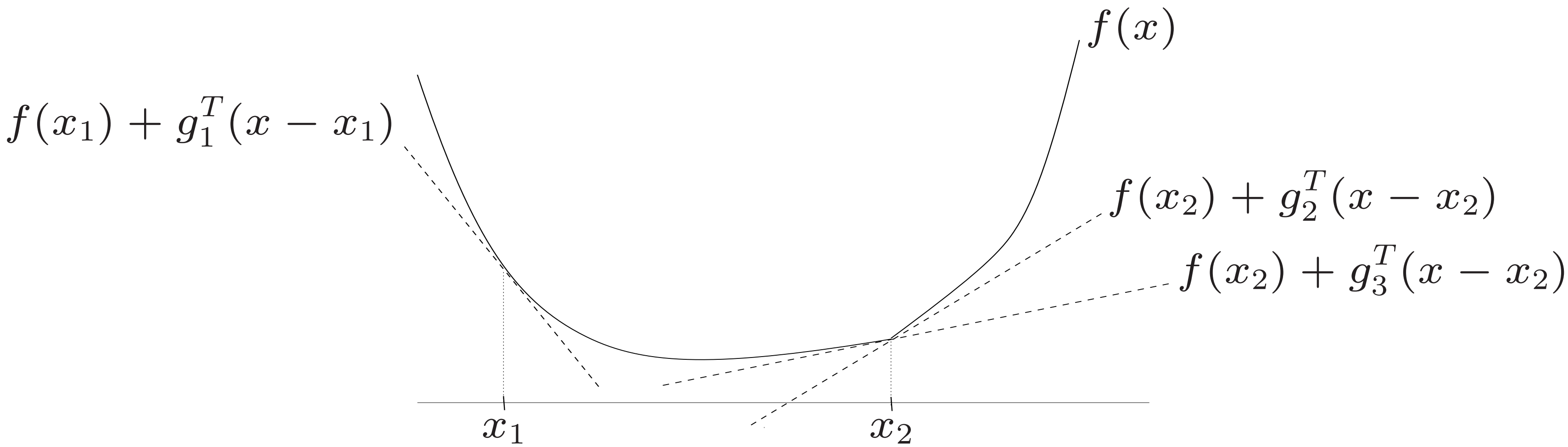
- variable metric methods
- conjugate gradient method
- accelerated proximal gradient method

we will discuss subgradient and proximal gradient methods

Subgradient

g is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \mathbf{dom}\, f$$



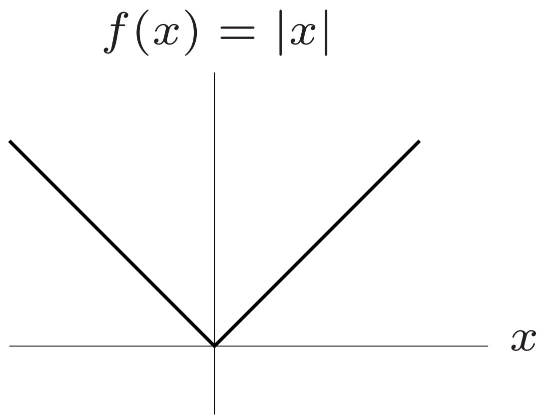
generalizes basic inequality for convex differentiable f

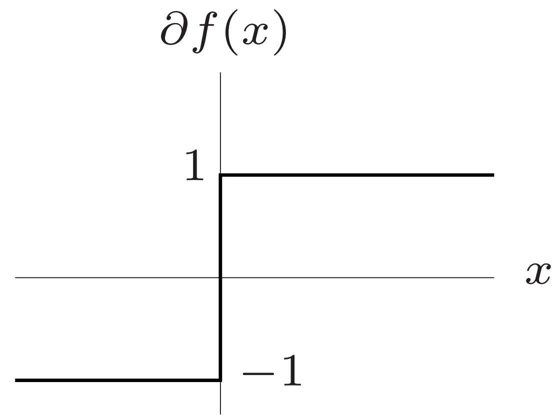
$$f(y) \geq f(x) + \nabla f(x)^T(y - x) \quad \forall y \in \mathbf{dom}\, f$$

Subdifferential

the set of all subgradients of f at x is called the **subdifferential** $\partial f(x)$

absolute value $f(x) = |x|$





Euclidean norm $f(x) = \|x\|_2$

$$\partial f(x) = \frac{1}{\|x\|_2} x \quad \text{if } x \neq 0, \qquad \partial f(x) = \{g \mid \|g\|_2 \leq 1\} \quad \text{if } x = 0$$

Subgradient calculus

weak calculus

rules for finding **one** subgradient

- sufficient for most algorithms for nondifferentiable convex optimization
- if one can evaluate $f(x)$, one can usually compute a subgradient
- much easier than finding the entire subdifferential

subdifferentiability

- convex f is subdifferentiable on $\mathbf{dom}\, f$ except possibly at the boundary
- example of a non-subdifferentiable function: $f(x) = -\sqrt{x}$ at $x = 0$

Examples of calculus rules

nonnegative combination: $f = \alpha_1 f_1 + \alpha_2 f_2$ with $\alpha_1, \alpha_2 \geq 0$

$$g = \alpha_1 g_1 + \alpha_2 g_2, \qquad g_1 \in \partial f_1(x), \quad g_2 \in \partial f_2(x)$$

composition with affine transformation: $f(x) = h(Ax + b)$

$$g = A^T \tilde{g}, \qquad \tilde{g} \in \partial h(Ax + b)$$

pointwise maximum $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$

$$g \in \partial f_i(x) \quad \text{where } f_i(x) = \max_k f_k(x)$$

conjugate $f^*(x) = \sup_y (x^T y - f(y))$: take any maximizing y

Subgradient method

to minimize a nondifferentiable convex function f: choose $x^{(0)}$ and repeat

$$x^{(k)} = x^{(k-1)} - t_k g^{(k-1)}, \quad k = 1, 2, \ldots$$

$g^{(k-1)}$ is **any** subgradient of f at $x^{(k-1)}$

step size rules

- fixed step size: t_k constant
- fixed step length: $t_k \|g^{(k-1)}\|_2$ constant (*i.e.*, $\|x^{(k)} - x^{(k-1)}\|_2$ constant)
- diminishing: $t_k \to 0$, $\sum_{k=1}^{\infty} t_k = \infty$

Some convergence results

assumption: f is convex and Lipschitz continuous with constant $G > 0$:

$$|f(x) - f(y)| \leq G\|x - y\|_2 \qquad \forall x, y$$

results

- fixed step size $t_k = t$

  converges to approximately $G^2 t/2$-suboptimal

- fixed length $t_k \|g^{(k-1)}\|_2 = s$

  converges to approximately $Gs/2$-suboptimal

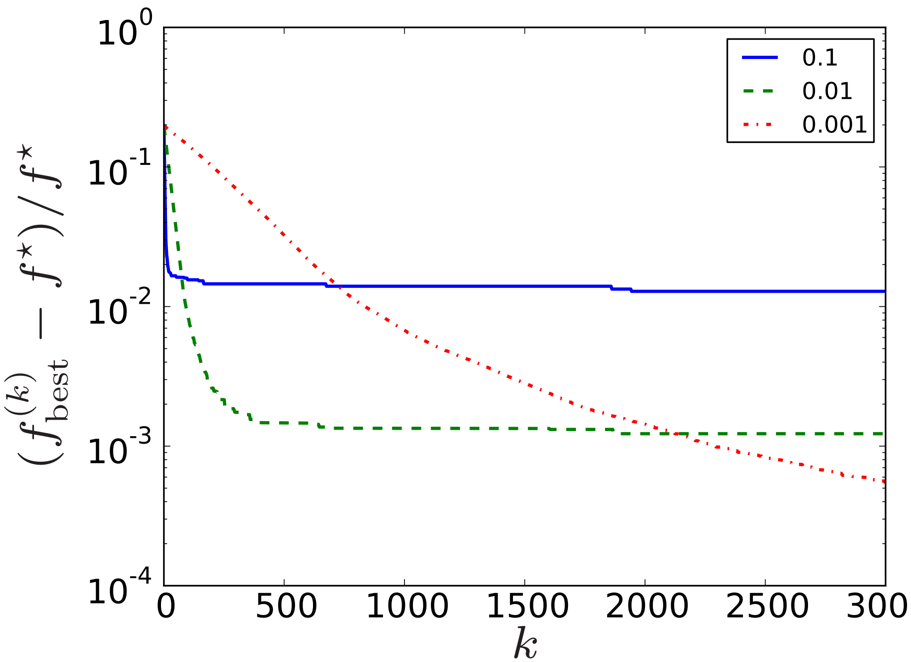
- decreasing $\sum_k t_k \to \infty$, $t_k \to 0$: convergence

  rate of convergence is $1/\sqrt{k}$ with proper choice of step size sequence

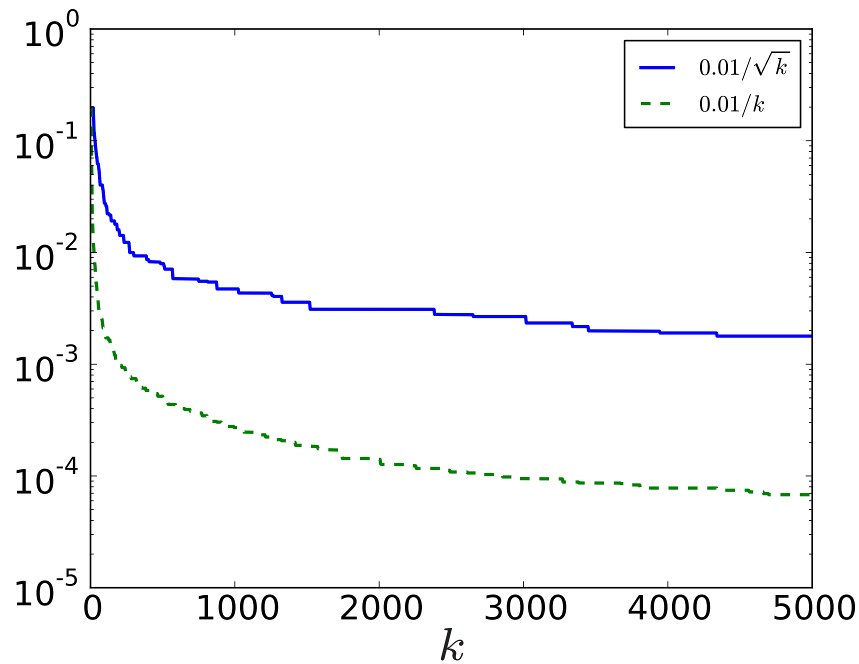
Example: 1-norm minimization

$$\text{minimize} \quad \|Ax - b\|_1 \qquad (A \in \mathbf{R}^{500\times 100}, b \in \mathbf{R}^{500})$$

subgradient is given by $A^T \mathbf{sign}(Ax - b)$



fixed steplength
$s = 0.1,\ 0.01,\ 0.001$



diminishing step size
$t_k = 0.01/\sqrt{k}$, $t_k = 0.01/k$

Outline

- gradient and subgradient method
- **proximal gradient method**
- fast proximal gradient methods

Proximal operator

the proximal operator (prox-operator) of a convex function h is

$$\mathbf{prox}_h(x) = \operatorname*{argmin}_u \left(h(u) + \frac{1}{2}\|u - x\|_2^2 \right)$$

- $h(x) = 0$: $\mathbf{prox}_h(x) = x$
- $h(x) = I_C(x)$ (indicator function of C): $\mathbf{prox}_h$ is projection on C

$$\mathbf{prox}_h(x) = \operatorname*{argmin}_{u \in C} \|u - x\|_2^2 = P_C(x)$$

- $h(x) = \|x\|_1$: $\mathbf{prox}_h$ is the 'soft-threshold' (shrinkage) operation

$$\mathbf{prox}_h(x)_i = \begin{cases} x_i - 1 & x_i \geq 1 \\ 0 & |x_i| \leq 1 \\ x_i + 1 & x_i \leq -1 \end{cases}$$

Proximal gradient method

unconstrained problem with cost function split in two components

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

- g convex, differentiable, with $\mathbf{dom}\, g = \mathbf{R}^n$
- h convex, possibly nondifferentiable, with inexpensive prox-operator

proximal gradient algorithm

$$x^{(k)} = \mathbf{prox}_{t_k h}\left(x^{(k-1)} - t_k \nabla g(x^{(k-1)})\right)$$

$t_k > 0$ is step size, constant or determined by line search

Examples

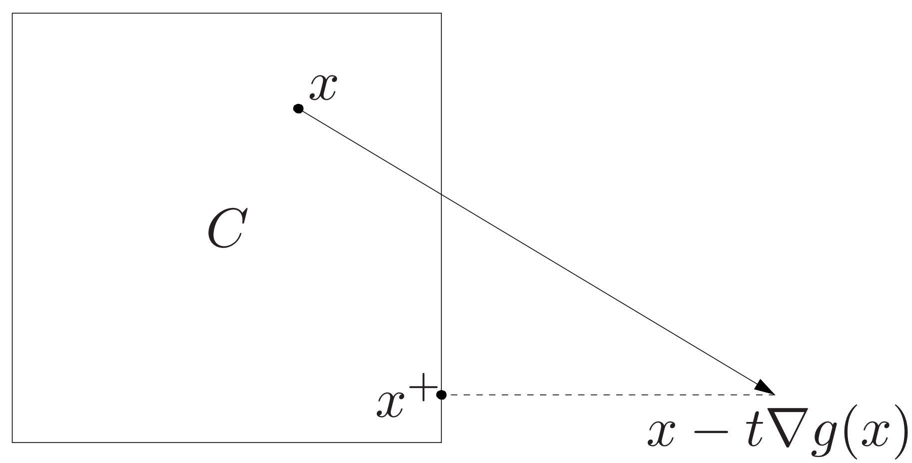
$$\text{minimize} \quad g(x) + h(x)$$

gradient method: $h(x) = 0$, *i.e.*, minimize $g(x)$

$$x^+ = x - t\nabla g(x)$$

gradient projection method: $h(x) = I_C(x)$, *i.e.*, minimize $g(x)$ over C

$$x^+ = P_C\left(x - t\nabla g(x)\right)$$

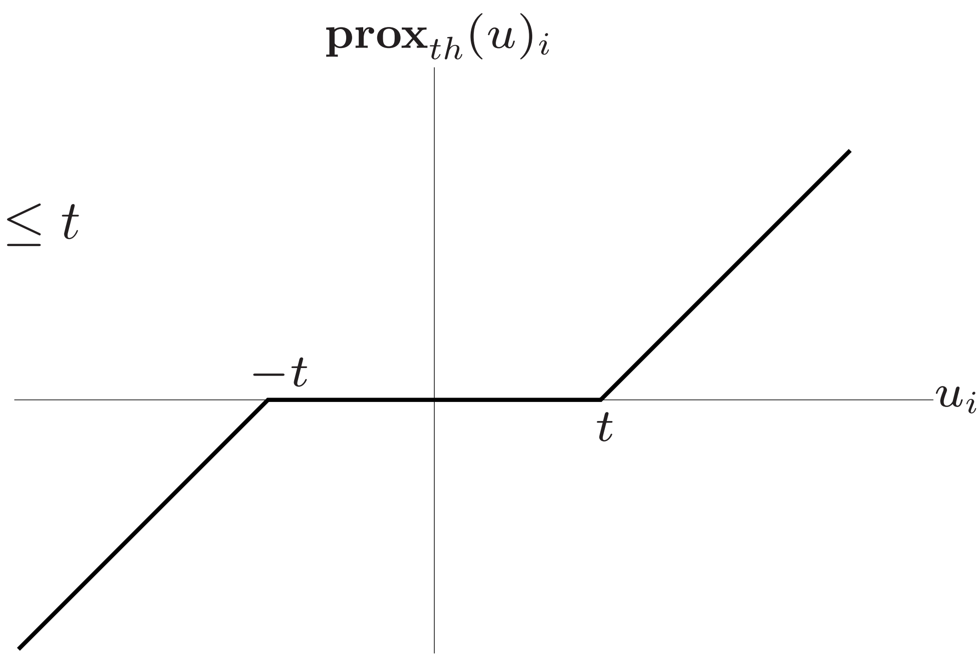


iterative soft-thresholding: $h(x) = \|x\|_1$

$$x^+ = \mathbf{prox}_{th}\left(x - t\nabla g(x)\right)$$

where

$$\mathbf{prox}_{th}(u)_i = \begin{cases} u_i - t & u_i \geq t \\ 0 & -t \leq u_i \leq t \\ u_i + t & u_i \leq -t \end{cases}$$



Properties of proximal operator

$$\mathbf{prox}_h(x) = \underset{u}{\operatorname{argmin}} \left(h(u) + \frac{1}{2}\|u - x\|_2^2 \right)$$

assume h is closed and convex (*i.e.*, convex with closed epigraph)

- $\mathbf{prox}_h(x)$ is uniquely defined for all x
- $\mathbf{prox}_h$ is nonexpansive

$$\|\mathbf{prox}_h(x) - \mathbf{prox}_h(y)\|_2 \leq \|x - y\|_2$$

- Moreau decomposition

$$x = \mathbf{prox}_h(x) + \mathbf{prox}_{h^*}(x)$$

Moreau-Yosida regularization

$$h_{(t)}(x) = \inf_u \left(h(u) + \frac{1}{2t}\|u - x\|_2^2 \right) \qquad \text{(with } t > 0\text{)}$$

- $h_{(t)}$ is convex (infimum over u of a convex function of x, u)
- domain of $h_{(t)}$ is $\mathbf{R}^n$ (minimizing $u = \mathbf{prox}_{th}(x)$ is defined for all x)
- $h_{(t)}$ is differentiable with gradient

$$\nabla h_{(t)}(x) = \frac{1}{t}\left(x - \mathbf{prox}_{th}(x)\right)$$

  gradient is Lipschitz continuous with constant $1/t$

- can interpret $\mathbf{prox}_{th}(x)$ as gradient step $x - t\nabla h_{(t)}(x)$

Examples

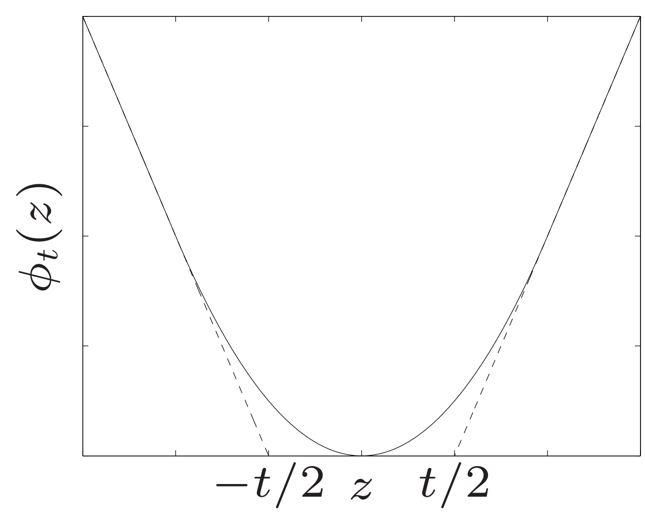
indicator function (of closed convex set C): squared Euclidean distance

$$h(x) = I_C(x), \qquad h_{(t)}(x) = \frac{1}{2t}\,\mathbf{dist}(x)^2$$

1-norm: Huber penalty

$$h(x) = \|x\|_1, \qquad h_{(t)}(x) = \sum_{k=1}^{n} \phi_t(x_k)$$

$$\phi_t(z) = \begin{cases} z^2/(2t) & |z| \le t \\ |z| - t/2 & |z| \ge t \end{cases}$$



Examples of inexpensive prox-operators

projection on simple sets

- hyperplanes and halfspaces
- rectangles
$$\{x \mid l \leq x \leq u\}$$
- probability simplex
$$\{x \mid \mathbf{1}^T x = 1, x \geq 0\}$$
- norm ball for many norms (Euclidean, 1-norm, . . .)
- nonnegative orthant, second-order cone, positive semidefinite cone

Euclidean norm: $h(x) = \|x\|_2$

$$\mathbf{prox}_{th}(x) = \left(1 - \frac{t}{\|x\|_2}\right) x \quad \text{if } \|x\|_2 \geq t, \qquad \mathbf{prox}_{th}(x) = 0 \quad \text{otherwise}$$

logarithmic barrier

$$h(x) = -\sum_{i=1}^{n} \log x_i, \qquad \mathbf{prox}_{th}(x)_i = \frac{x_i + \sqrt{x_i^2 + 4t}}{2}, \quad i = 1, \ldots, n$$

Euclidean distance: $d(x) = \inf_{y \in C} \|x - y\|_2$ (C closed convex)

$$\mathbf{prox}_{td}(x) = \theta P_C(x) + (1 - \theta)x, \qquad \theta = \frac{t}{\max\{d(x), t\}}$$

generalizes soft-thresholding operator

Prox-operator of conjugate

$$\mathbf{prox}_{th}(x) = x - t\,\mathbf{prox}_{h^*/t}(x/t)$$

- follows from Moreau decomposition
- of interest when prox-operator of h^* is inexpensive

example: norms

$$h(x) = \|x\|, \qquad h^*(y) = I_C(y)$$

where C is unit ball for dual norm $\|\cdot\|_*$

- $\mathbf{prox}_{h*/t}$ is projection on C
- formula useful for prox-operator of $\|\cdot\|$ if projection on C is inexpensive

Support function

many convex functions can be expressed as **support functions**

$$h(x) = S_C(x) = \sup_{y \in C} x^T y$$

with C closed, convex

- conjugate is indicator function of C: $h^*(y) = I_C(y)$
- hence, can compute $\mathbf{prox}_{th}$ via projection on C

example: $h(x)$ is sum of largest r components of x

$$h(x) = x_{[1]} + \cdots + x_{[r]} = S_C(x), \qquad C = \{y \mid 0 \le y \le \mathbf{1}, \mathbf{1}^T y = r\}$$

Convergence of proximal gradient method

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

assumptions

- ∇g is Lipschitz continuous with constant $L > 0$

$$\|\nabla g(x) - \nabla g(y)\|_2 \leq L\|x - y\|_2 \quad \forall x, y$$

- optimal value $f^\star$ is finite and attained at $x^\star$ (not necessarily unique)

result: with fixed step size $t_k = 1/L$

$$f(x^{(k)}) - f^\star \leq \frac{L}{2k}\|x^{(0)} - x^\star\|_2^2$$

- compare with $1/\sqrt{k}$ rate of subgradient method
- can be extended to include line searches

Outline

- gradient and subgradient method
- proximal gradient method
- **fast proximal gradient methods**

Fast (proximal) gradient methods

- Nesterov (1983, 1988, 2005): three gradient projection methods with $1/k^2$ convergence rate
- Beck & Teboulle (2008): FISTA, a proximal gradient version of Nesterov's 1983 method
- Nesterov (2004 book), Tseng (2008): overview and unified analysis of fast gradient methods
- several recent variations and extensions

this lecture: FISTA (Fast Iterative Shrinkage-Thresholding Algorithm)

FISTA

unconstrained problem with composite objective

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

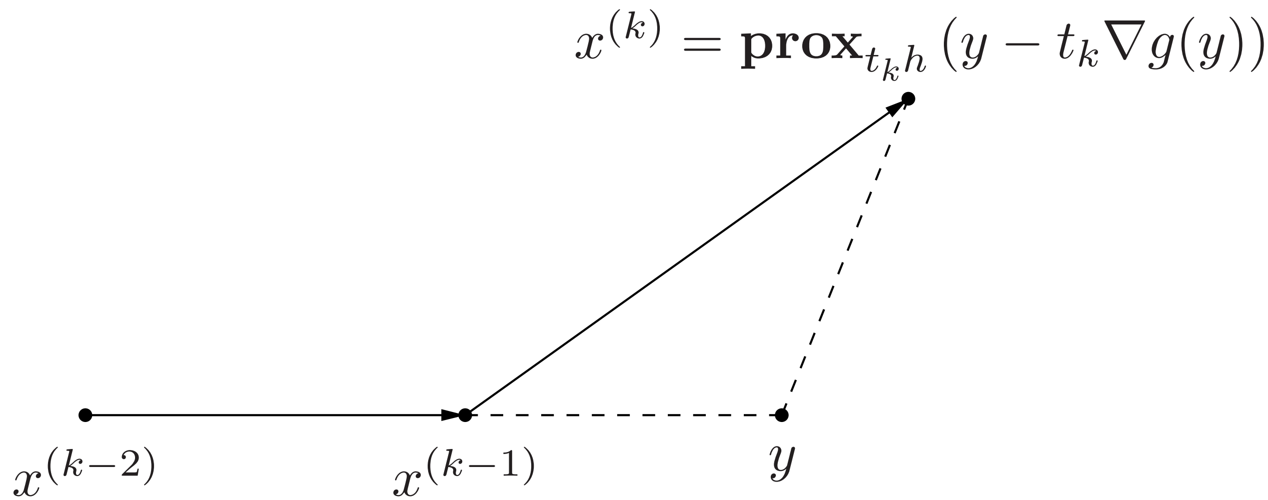
- g convex differentiable with $\mathbf{dom}\, g = \mathbf{R}^n$
- h convex with inexpensive prox-operator

algorithm: choose any $x^{(0)} = x^{(-1)}$; for $k \geq 1$, repeat the steps

$$\begin{aligned} y &= x^{(k-1)} + \frac{k-2}{k+1}\left(x^{(k-1)} - x^{(k-2)}\right) \\ x^{(k)} &= \mathbf{prox}_{t_k h}\left(y - t_k \nabla g(y)\right) \end{aligned}$$

Interpretation

- first two iterations ($k = 1, 2$) are proximal gradient steps at $x^{(k-1)}$
- next iterations are proximal gradient steps at extrapolated points y



sequence $x^{(k)}$ remains feasible (in $\mathbf{dom}\, h$); y may be outside $\mathbf{dom}\, h$

Convergence of FISTA

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

assumptions

- $\mathbf{dom}\, g = \mathbf{R}^n$ and ∇g is Lipschitz continuous with constant $L > 0$
- h is closed (implies $\mathbf{prox}_{th}(u)$ exists and is unique for all u)
- optimal value $f^\star$ is finite and attained at $x^\star$ (not necessarily unique)

result: with fixed step size $t_k = 1/L$

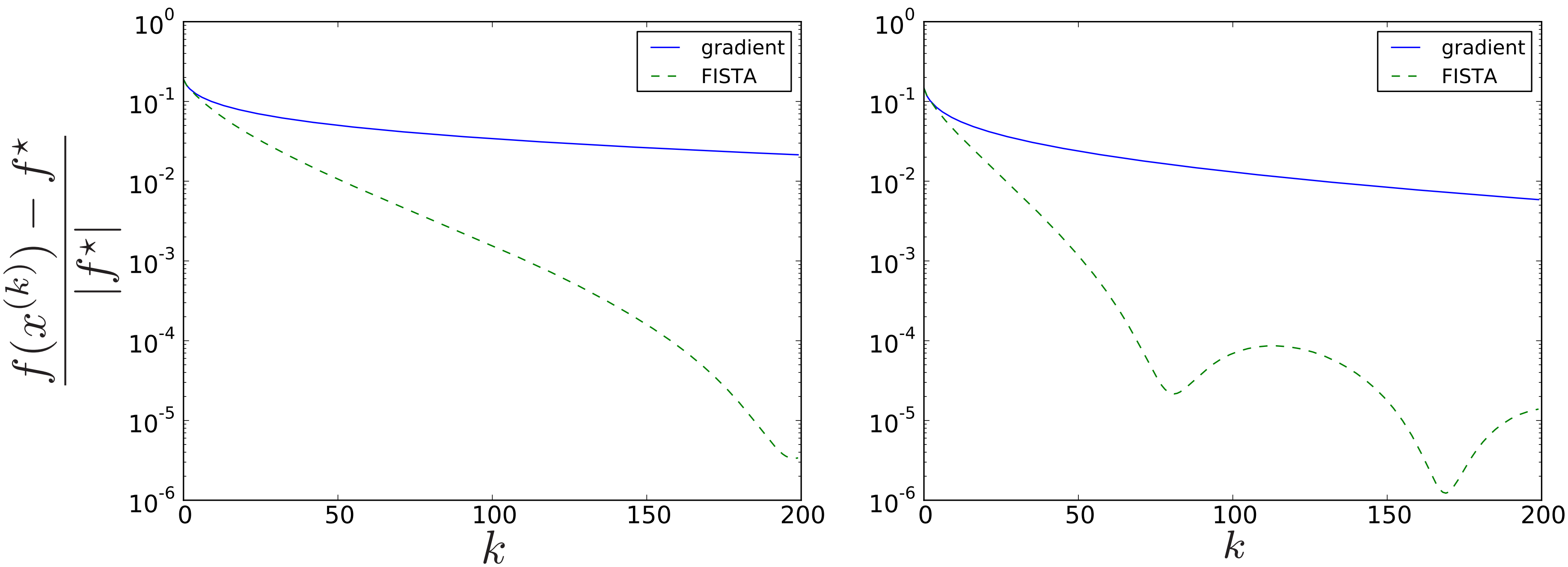
$$f(x^{(k)}) - f^\star \le \frac{2L}{(k+1)^2} \|x^{(0)} - f^\star\|_2^2$$

- compare with $1/k$ convergence rate for gradient method
- can be extended to include line searches

Example

$$\text{minimize} \quad \log \sum_{i=1}^{m} \exp(a_i^T x + b_i)$$

randomly generated data with $m = 2000$, $n = 1000$, same fixed step size



FISTA is not a descent method

Dual methods

- Lagrange duality
- dual decomposition
- dual proximal gradient method
- multiplier methods

Dual function

convex problem (with linear constraints for simplicity)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

Lagrangian

$$L(x, \lambda, \nu) = f(x) + \lambda^T(Gx - h) + \nu^T(Ax - b)$$

dual function

$$\begin{aligned} g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\ &= -f^*(-G^T\lambda - A^T\nu) - h^T\lambda - b^T\nu \end{aligned}$$

$f^*(y) = \sup_x(y^Tx - f(x))$ is conjugate of f

Dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

a convex optimization problem in λ, ν

duality theorem ($p^\star$ is primal optimal value, $d^\star$ is dual optimal value)

- weak duality: $p^\star \geq d^\star$ (without exception)
- strong duality: $p^\star = d^\star$ if a constraint qualification holds

  (for example, primal problem is feasible and $\mathbf{dom}\, f$ open)

Norm approximation

$$\text{minimize} \quad \|Ax - b\|$$

reformulated problem

$$\begin{array}{ll} \text{minimize} & \|y\| \\ \text{subject to} & y = Ax - b \end{array}$$

dual function

$$\begin{aligned} g(\nu) &= \inf_{x,y} \left(\|y\| + \nu^T y - \nu^T A x + b^T \nu \right) \\ &= \begin{cases} b^T \nu & A^T \nu = 0, \quad \|\nu\|_* \le 1 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

dual problem

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z = 0, \quad \|z\|_* \le 1 \end{array}$$

Karush-Kuhn-Tucker optimality conditions

if strong duality holds, then x, λ, ν are optimal if and only if

1. x is *primal feasible*

$$x \in \mathbf{dom}\, f, \qquad Gx \preceq h, \qquad Ax = b$$

2. $\lambda \succeq 0$

3. *complementary slackness* holds

$$\lambda^T(h - Gx) = 0$$

4. x minimizes $L(x, \lambda, \nu) = f(x) + \lambda^T(Gx - h) + \nu^T(Ax - b)$

   for differentiable f, condition 4 can be expressed as

$$\nabla f(x) + G^T\lambda + A^T\nu = 0$$

Outline

- Lagrange dual
- **dual decomposition**
- dual proximal gradient method
- multiplier methods

Dual methods

primal problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - b^T\nu - f^*(-G^T\lambda - A^T\nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

possible advantages of solving the dual when using first-order methods

- dual problem is unconstrained or has simple constraints
- dual is differentiable
- dual (almost) decomposes into smaller problems

(Sub-)gradients of conjugate function

$$f^*(y) = \sup_x \left(y^T x - f(x) \right)$$

- subgradient: x is a subgradient at y if it maximizes $y^T x - f(x)$
- if maximizing x is unique, then f^* is differentiable at y

  this is the case, for example, if f is strictly convex

strongly convex function: f is strongly convex with modulus $\mu > 0$ if

$$f(x) - \frac{\mu}{2} x^T x \quad \text{is convex}$$

implies that $\nabla f^*(x)$ is Lipschitz continuous with parameter $1/\mu$

Dual gradient method

primal problem with equality constraints and dual

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

dual ascent: use (sub-)gradient method to minimize

$$-g(\nu) = b^T\nu + f^*(-A^T\nu) = \sup_x \left((b - Ax)^T\nu - f(x)\right)$$

algorithm

$$\begin{aligned} x &= \operatorname*{argmin}_{\hat{x}} \left(f(\hat{x}) + \nu^T(A\hat{x} - b)\right) \\ \nu^+ &= \nu + t(Ax - b) \end{aligned}$$

of interest if calculation of x is inexpensive (for example, separable)

Dual decomposition

convex problem with separable objective, coupling constraints

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & G_1x_1 + G_2x_2 \leq h \end{array}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - f_1^*(-G_1^T\lambda) - f_2^*(-G_2^T\lambda) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

- can be solved by (sub-)gradient projection if $\lambda \geq 0$ is the only constraint
- evaluating objective involves two independent minimizations

$$f_j^*(-G_j^T\lambda) = -\inf_{x_j} \left(f_j(x_j) + \lambda^T G_j x_j \right)$$

minimizer x_j gives subgradient $-G_jx_j$ of $f_j^*(-G_j^T\lambda)$ with respect to λ

dual subgradient projection method

- solve two unconstrained (and independent) subproblems

$$x_j = \underset{\hat{x}_j}{\operatorname{argmin}} \left(f_j(\hat{x}_j) + \lambda^T G_j \hat{x}_j \right), \quad j = 1, 2$$

- make projected subgradient update of λ

$$\lambda^+ = (\lambda + t(G_1 x_1 + G_2 x_2 - h))_+$$

interpretation: price coordination between two units in a system

- constraints are limits on shared resources; λ_i is price of resource i
- dual update $\lambda_i^+ = (\lambda_i - t s_i)_+$ depends on slacks $s = h - G_1 x_1 - G_2 x_2$
  - increases price λ_i if resource is over-utilized ($s_i < 0$)
  - decreases price λ_i if resource is under-utilized ($s_i > 0$)
  - never lets prices get negative

Outline

- Lagrange dual
- dual decomposition
- **dual proximal gradient method**
- multiplier methods

First-order dual methods

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Gx \geq h \\ & Ax = b \end{array} \qquad \begin{array}{ll} \text{maximize} & -f^*(-G^T\lambda - A^T\nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

subgradient method: slow, step size selection difficult

gradient method: faster, requires differentiable f^*

- in many applications f^* is not differentiable, or has nontrivial domain
- f^* can be smoothed by adding a small strongly convex term to f

proximal gradient method (this section): dual cost split in two terms

- first term is differentiable
- second term has an inexpensive prox-operator

Composite structure in the dual

primal problem with separable objective

$$\begin{array}{ll}\text{minimize} & f(x) + h(y) \\ \text{subject to} & Ax + By = b\end{array}$$

dual problem

$$\text{maximize} \quad -f^*(A^T z) - h^*(B^T z) + b^T z$$

has the composite structure required for the proximal gradient method if

- f is strongly convex; hence ∇f^* is Lipschitz continuous
- prox-operator of $h^*(B^T z)$ is cheap (closed form or simple algorithm)

Regularized norm approximation

$$\text{minimize} \quad f(x) + \|Ax - b\|$$

f strongly convex with modulus μ; $\|\cdot\|$ is any norm

reformulated problem and dual

$$\begin{array}{ll} \text{minimize} & f(x) + \|y\| \\ \text{subject to} & y = Ax - b \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & b^T z - f^*(A^T z) \\ \text{subject to} & \|z\|_* \leq 1 \end{array}$$

- gradient of dual cost is Lipschitz continuous with parameter $\|A\|_2^2/\mu$

$$\nabla f^*(A^T z) = \operatorname*{argmin}_x \left(f(x) - z^T A x \right)$$

- for most norms, projection on dual norm ball is inexpensive

dual gradient projection algorithm for

$$\text{minimize} \quad f(x) + \|Ax - b\|$$

choose initial z and repeat

$$\begin{aligned} x &= \operatorname*{argmin}_{\hat{x}} \left(f(\hat{x}) - z^T A\hat{x} \right) \\ z^+ &= P_C\left(z + t(b - Ax)\right) \end{aligned}$$

- P_C is projection on $C = \{y \mid \|y\|_* \leq 1\}$
- step size t is constant or from backtracking line search
- can use accelerated gradient projection algorithm (FISTA) for z-update
- first step decouples if f is separable

Outline

- Lagrange dual
- dual decomposition
- dual proximal gradient method
- **multiplier methods**

Moreau-Yosida smoothing of the dual

dual of equality constrained problem

$$\text{maximize} \quad g(\nu) = \inf_x \left(f(x) + \nu^T(Ax - b) \right)$$

smoothed dual problem

$$\text{maximize} \quad g_{(t)}(\nu) = \sup_z \left(g(z) - \frac{1}{2t}\|z - \nu\|_2^2 \right)$$

- same solution as non-smoothed dual
- equivalent expression (from duality)

$$g_{(t)}(\nu) = \inf_x \left(f(x) + \nu^T(Ax - b) + \frac{t}{2}\|Ax - b\|_2^2 \right)$$

$\nabla g_{(t)}(\nu) = Ax - b$ with x the minimizer in the definition

Augmented Lagrangian method

algorithm: choose initial ν and repeat

$$\begin{aligned} x &= \operatorname*{argmin}_{\hat{x}} L_t(\hat{x}, \nu) \\ \nu^+ &= \nu + t(Ax - b) \end{aligned}$$

- L_t is the augmented Lagrangian (Lagrangian plus quadratic penalty)

$$L_t(x, \nu) = f(x) + \nu^T(Ax - b) + \frac{t}{2}\|Ax - b\|_2^2$$

- maximizes smoothed dual function g_t via gradient method
- can be extended to problems with inequality constraints

Dual decomposition

convex problem with separable objective

$$\begin{array}{ll} \text{minimize} & f(x) + h(y) \\ \text{subject to} & Ax + By = b \end{array}$$

augmented Lagrangian

$$L_t(x, y, \nu) = f(x) + h(y) + \nu^T(Ax + By - b) + \frac{t}{2}\|Ax + By - b\|_2^2$$

- difficulty: quadratic penalty destroys separability of Lagrangian
- solution: replace minimization over (x, y) by alternating minimization

Alternating direction method of multipliers

apply one cycle of alternating minimization steps to augmented Lagrangian

1. minimize augmented Lagrangian over x:

$$x^{(k)} = \operatorname*{argmin}_x L_t(x, y^{(k-1)}, \nu^{(k-1)})$$

2. minimize augmented Lagrangian over y:

$$y^{(k)} = \operatorname*{argmin}_y L_t(x^{(k)}, y, \nu^{(k-1)})$$

3. dual update:

$$\nu^{(k)} := \nu^{(k-1)} + t\left(Ax^{(k)} + By^{(k)} - b\right)$$

can be shown to converge under weak assumptions

Example: regularized norm approximation

$$\text{minimize} \quad f(x) + \|Ax - b\|$$

f convex (not necessarily strongly)

reformulated problem

$$\begin{array}{ll} \text{minimize} & f(x) + \|y\| \\ \text{subject to} & y = Ax - b \end{array}$$

augmented Lagrangian

$$L_t(x, y, z) = f(x) + \|y\| + z^T(y - Ax + b) + \frac{t}{2}\|y - Ax + b\|_2^2$$

ADMM steps (with $f(x) = \|x - a\|_2^2/2$ as example)

$$L_t(x, y, z) = f(x) + \|y\| + z^T(y - Ax + b) + \frac{t}{2}\|y - Ax + b\|_2^2$$

1. minimization over x

$$x := \operatorname*{argmin}_{\hat{x}} L_t(\hat{x}, y, \nu) = (I + tA^TA)^{-1}(a + A^T(z + t(y - b))$$

2. minimization over y via prox-operator of $\|\cdot\|/t$

$$y := \operatorname*{argmin}_{\hat{y}} L_t(x, \hat{y}, z) = \mathbf{prox}_{\|\cdot\|/t}(Ax - b - (1/t)z)$$

   can be evaluated via projection on dual norm ball $C = \{u \mid \|u\|_* \leq 1\}$

3. dual update: $z := z + t(y - Ax - b)$

cost per iteration dominated by linear equation in step 1

Example: sparse covariance selection

$$\text{minimize} \quad \mathbf{tr}(CX) - \log\det X + \|X\|_1$$

variable $X \in \mathbf{S}^n$; $\|X\|_1$ is sum of absolute values of X

reformulation

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) - \log\det X + \|Y\|_1 \\ \text{subject to} & X - Y = 0 \end{array}$$

augmented Lagrangian

$$\begin{aligned} & L_t(X, Y, Z) \\ & \quad = \quad \mathbf{tr}(CX) - \log\det X + \|Y\|_1 + \mathbf{tr}(Z(X - Y)) + \frac{t}{2}\|X - Y\|_F^2 \end{aligned}$$

ADMM steps: alternating minimization of augmented Lagrangian

$$\mathbf{tr}(CX) - \log\det X + \|Y\|_1 + \mathbf{tr}(Z(X-Y)) + \frac{t}{2}\|X-Y\|_F^2$$

- minimization over X:

$$X := \operatorname*{argmin}_{\hat{X}} \left(-\log\det \hat{X} + \frac{t}{2}\|\hat{X} - Y + \frac{1}{t}(C+Z)\|_F^2 \right)$$

  solution follows from eigenvalue decomposition of $Y - (1/t)(C+Z)$

- minimization over Y:

$$Y := \operatorname*{argmin}_{\hat{Y}} \left(\|\hat{Y}\|_1 + \frac{t}{2}\|\hat{Y} - X - \frac{1}{t}Z\|_F^2 \right)$$

  apply element-wise soft-thresholding to $X - (1/t)Z$

- dual update $Z := Z + t(X-Y)$

cost per iteration dominated by cost of eigenvalue decomposition

Douglas-Rachford splitting algorithm

$$\text{minimize} \quad g(x) + h(x)$$

with g and h closed convex functions

algorithm

$$\begin{aligned}
\hat{x}^{(k+1)} &= \mathbf{prox}_{tg}(x^{(k)} - y^{(k)}) \\
x^{(k+1)} &= \mathbf{prox}_{th}(\hat{x}^{(k+1)} + y^{(k)}) \\
y^{(k+1)} &= y^{(k)} + \hat{x}^{(k+1)} - x^{(k+1)}
\end{aligned}$$

- converges under weak conditions (existence of a solution)
- useful when g and h have inexpensive prox-operators

ADMM as Douglas-Rachford algorithm

$$\begin{array}{ll} \text{minimize} & f(x) + h(y) \\ \text{subject to} & Ax + By = b \end{array}$$

dual problem

$$\text{maximize} \quad b^T z - f^*(A^T z) - h^*(B^T z)$$

ADMM algorithm

- split dual objective in two terms $g_1(z) + g_2(z)$

$$g_1(z) = b^T z - f^*(A^T z), \qquad g_2(z) - h^*(B^T z)$$

- Douglas-Rachford algorithm applied to the dual gives ADMM

Sources and references

these lectures are based on the courses

- EE364A (S. Boyd, Stanford), EE236B (UCLA), *Convex Optimization*

  `www.stanford.edu/class/ee364a`
  `www.ee.ucla.edu/ee236b/`

- EE236C (UCLA) *Optimization Methods for Large-Scale Systems*

  `www.ee.ucla.edu/~vandenbe/ee236c`

- EE364B (S. Boyd, Stanford University) *Convex Optimization II*

  `www.stanford.edu/class/ee364b`

see the websites for expanded notes, references to literature and software