CEE 142L

Reinforced Concrete Structures
Laboratory

Beam Experiment

Material Models

Steel

In the analysis of the beam specimens, material property information is needed to accurately predict their behavior. Certain quantities will be given, i.e. steel yield stress (f_y) and specified compressive strength of concrete (f'_c), others must be related to this given data and known material relations.

Figure 1 shows a typical stress – strain curve for steel. There are three important regions labeled in the figure.

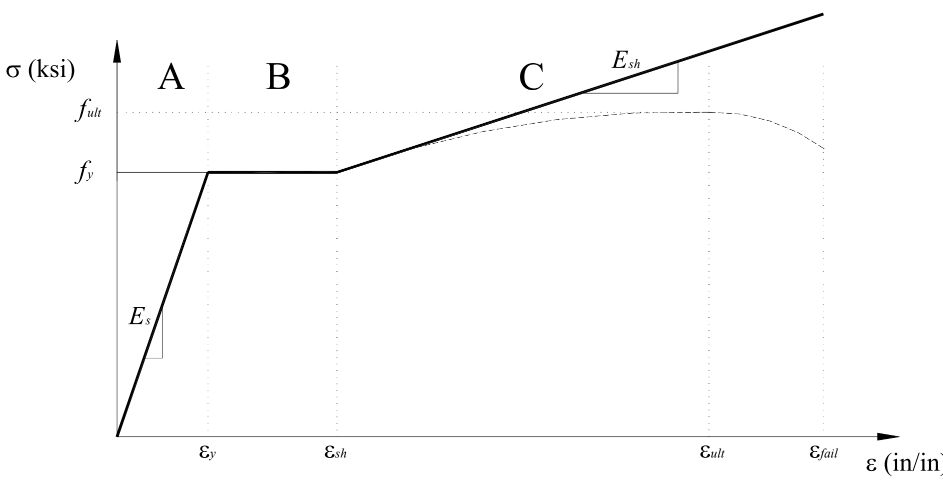


Figure 1 - Stress-strain relation for steel

Region 'A' is the linear–elastic zone. Stress is directly proportional to strain. This proportion is Young's modulus (E_s) for steel and is equal to 29,000,000 pounds per square inch (psi). The upper limit of this region has coordinates (f_y, ε_y), where ε_y is the yield strain, equal to f_y /E_s.

Region 'B' is a stress plateau, where steel will deform without application of additional load. Between strains of ε_y and ε_{sh} (the strain at the onset of strain hardening) the steel stress is equal to f_y.

Region 'C' is the strain-hardening portion of the stress–strain curve. As the strain is increased beyond ε_{sh}, the stress increases moderately (shown in figure 1 as dotted line). For analysis, it is common to assume a linear stress-strain relation in the strain-hardening region (solid line in figure 1). Typically, a strain hardening modulus, E_{sh}, is taken equal to $E_s/30$ for f_y = 60 kips per square inch (ksi) and $E_s/50$ for f_y = 40 ksi. The strain at the onset of strain hardening, ε_{sh} is also dependant on the yield stress of steel.

Concrete

Figure 2 depicts the stress-strain relationship for concrete. While there are several analytical models that represent the stress-strain curve, the Hognestad model will be used in this class. This model is comprised of two regions.

Region 'A' depicts a parabolic relationship between stress and strain. This relation is defined as:

$$f = f_c^{'}\left[\frac{\varepsilon}{\varepsilon_o} + \left(\frac{\varepsilon}{2\varepsilon_o}\right)^2\right]$$

where ε_o is the strain at the peak stress. It typically has a value between 0.0015 to 0.003 in/in. For this experiment, it will be taken as 0.002.

Region 'B' lies beyond ε_o and represents a linearly decreasing stress-strain relation. At a strain of 0.0035, the model gives the stress to be $0.85f'_c$. Using this, the function of the line can be determined and is defined by:

$$f = f_c^{'}\left(1 - 0.15\frac{\varepsilon - \varepsilon_o}{0.0035 - \varepsilon_o}\right)$$

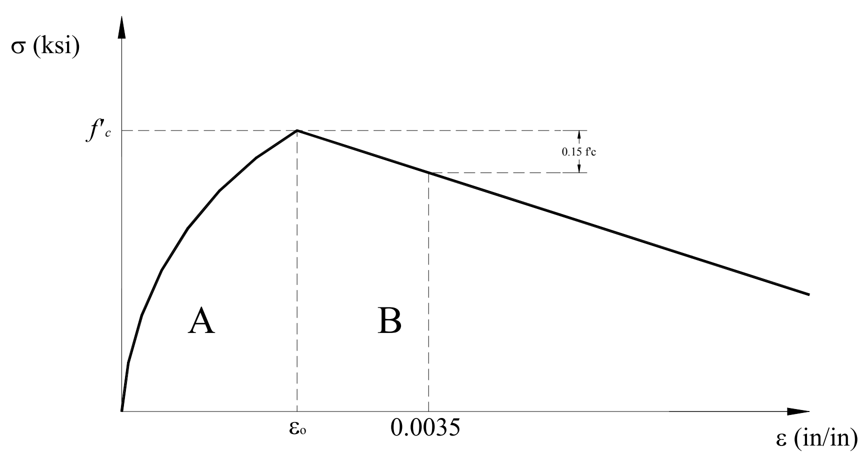


Figure 2 - Stress-strain relationship for concrete

Flexural Response of Beam Specimens

Using the material models defined earlier, the flexural response of the beams can now be presented. This response is used to define the Moment-Curvature (M-Φ) and Load-Displacement (P-Δ) diagrams. These diagrams depict the behavior of a beam as well as the anticipated failure limits.

There are a few underlying assumptions in flexural theory that should be understood before making any calculations. These are:

- Plane sections remain plane, that is, sections perpendicular to the axis of bending, which are plane, remain plane after bending.
- The strains in the concrete and in the steel are the same at the same point. This assumes that there is a perfect bond between the steel and concrete.
- The stresses in the steel and the concrete can be computed from analytical models.
- The tensile strength of concrete is neglected in flexural strength calculations.
- The compressive stress-strain relationship for concrete must be in agreement with ACI Sec. 10.2.6.

Although any point on the M-Φ or P-Δ diagrams can be computed, only six will be calculated in this experiment to assess expected M-Φ and P-Δ responses. The points are connected to construct the diagrams. These points are representative of certain states occurring in the beam: immediately before cracking, yielding, ultimate ($\varepsilon_c = 0.003$) and 2 points where $\varepsilon_c > 0.003$. The point immediately after cracking is also useful and is very simple to determine. It is dependant on the point just before cracking and the yield point and will be discussed later.

Cracking

Before cracking, concrete is assumed to resist tensile force and strength contribution by any reinforcement bars (rebar) may be neglected. Concrete can no longer carry tension once it reaches the rupture stress (f_r), where f_r is:

$$f_r = 7.5\sqrt{f_c^{'}}$$

It is also acceptable to approximate f_r to be equal to 0.1 f'_c. The cracking moment, M_{cr} is:

$$M_{cr} = \frac{f_r I_g}{c}$$

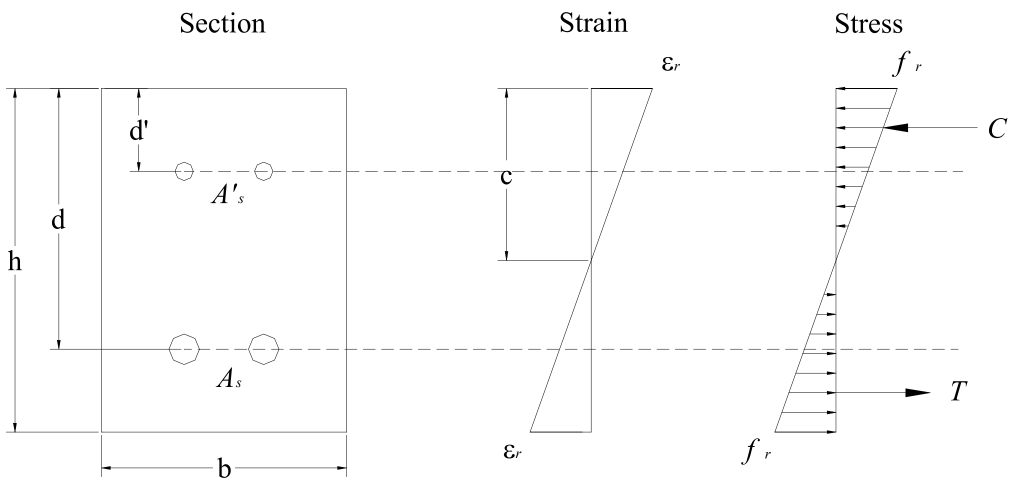


Figure 3 - Strain-stress plot just prior to cracking

Since the steel is neglected, the distance from the extreme compressive fiber to the neutral axis (c) is half of the beam depth, h. I_g is the gross moment of inertia of the beam cross-section. It is more accurate to include the steel in the moment of inertia and centroid calculations: however this typically has very little influence on M_{cr} and Φ_{cr} . To include the rebar, it is first transformed into an equivalent area of concrete by using the ratio of the material moduli. The moment of inertia is then calculated using the parallel axis theorem. The curvature (the slope of the strain diagram over cross-section depth) just prior to cracking is equal to:

$$\Phi_{cr} = \frac{f_r}{E_c c}$$

The stiffness of the section at the cracking moment is the slope of the M-Φ diagram between the origin and the cracking curvature (equal to M_{cr}/Φ_{cr}). E_c is the modulus of concrete and is given as:

$$E_c = 57000\sqrt{f_c^{'}}\,(psi)$$

The deflection (Δ_{cr}) is determined by inspecting the moment diagram at the point of interest. Any of several methods can be used to calculate the deflection, including double integration method or virtual work. The load (P_{cr}) necessary to cause this condition can then be calculated from the deflection. For a centrally loaded, simply-beam:

$$P_{cr} = \frac{4M_{cr}}{L}$$

Figure 3 shows the cross-section and corresponding strain and stress profiles for a beam just before cracking.

Immediately After Cracking

The moment capacity immediately after cracking is virtually the same as just before cracking, however the curvature is changed. No calculations are necessary to determine this point. First, a line is extended from the origin to the yield point on the M-Φ diagram (1). Next, a line is drawn from the point immediately before cracking horizontally to the right until it intersects with the line just drawn (2). This intersection represents the condition in the beam cross-section just after cracking (see figure 4).

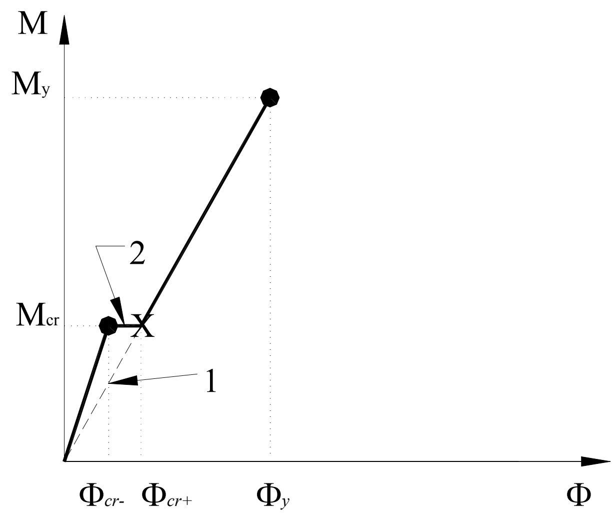


Figure 4- Curvature just after cracking

Yielding: $\varepsilon_s = \varepsilon_y$

The next point of interest on the M-Φ relation of a a beam is when the tension steelyields. ACI 318-99 requires that tension steel yield before concrete crushes in compression. This helps promote a ductile failure rather than a sudden brittle failure. However, keep in mind that the laboratory that beam specimens are not necessarily designed to satisfy ACI 318-99 requirements and may experience concrete crushing before tensile steel yielding.

Since yielding occurs after cracking, concrete is assumed to resist no tensile force. Rebar is now included in the strength calculations. Since the tensile steel is just yielding, the strain in the tension steel is:

$$\varepsilon_y = \frac{f_y}{E_s}$$

The strain in the compression steel and in the concrete can be determined using similar triangles. The distance from the compression surface to the neutral axis (ý) is required to define the strain gradient (or curvature) for the section. Using ý as an unknown, an expression of equilibrium is determined. This will result in a quadratic expression from which ý can be solved. Figure 5 shows the cross-section and corresponding strain and stress profiles for a beam at yielding.

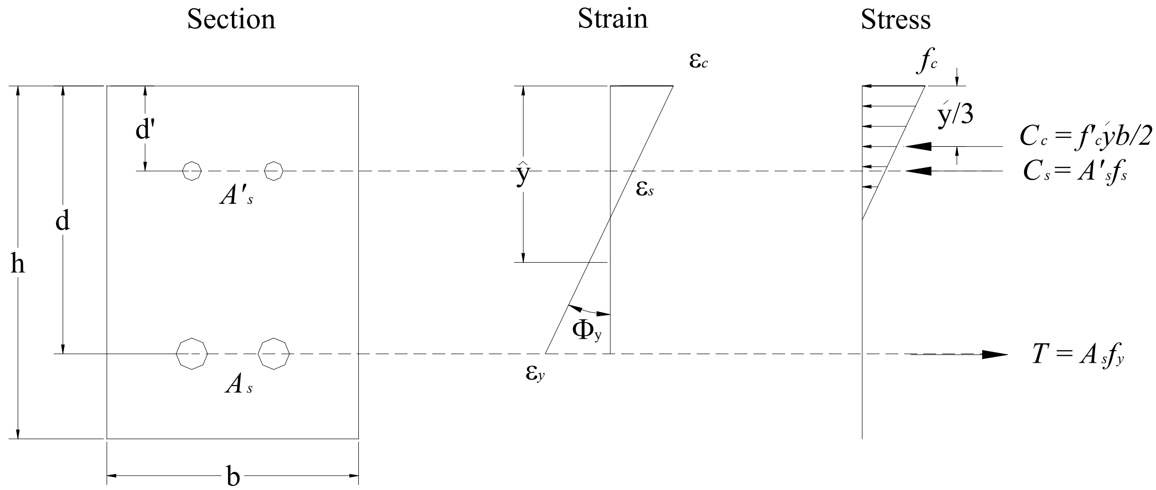


Figure 5 - Strain-stress plot at yielding

To determine the yield moment, M_y, use a relation similar to the one used in calculating M_{cr}:

$$M_y = \frac{\sigma_y I_{cr}}{d - \hat{y}} = \frac{f_y E_c I_{cr}}{E_s(d - \bar{y})}$$

I_{cr} for a rectangular section with compression and tension steel is equal to:

$$I_y = \frac{b\bar{y}^3}{3} + \frac{E_c A_s (\bar{y} - d)^2}{E_s} + \frac{E_c A_s^{'} (\bar{y} - d')^2}{E_s}$$

The curvature at yielding is equal to:

$$\Phi_y = \frac{\varepsilon_y}{d - \hat{y}}$$

The deflection (Δ_y) and loading (P_y) is determined in the same manner as just before cracking. However, different *EI* values must be used depending on whether the beam is cracked or not in different sections. Figure 6 illustrates this more clearly. The moment area method is the easiest method to implement this.

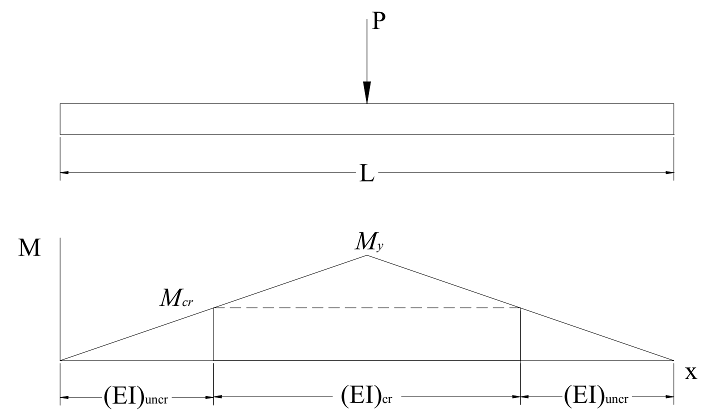


Figure 6 - Moment diagram of centrally loaded beam at yielding

ACI 318 Nominal Capacity: ε_c = 0.003

At the ultimate loading condition, the beam is assumed to reach a strain of 0.003 at the extreme fiber. A Whitney stress block of intensity $0.85f'_c$ and depth a is used to approximate the stress distribution in the compression zone. The depth a is equal to $\beta_1 x$. x is the distance from the outermost compressive fiber to the neutral axis. β_1 is a factor defined in ACI 318-99 Sec. 10.2.7.3 as:

$$\beta_1 = 0.85 \text{ for } f_c^{'} \leq 4000 \text{ psi}$$
$$0.85 \geq \beta_1 \geq 0.65 \text{ for } 4000 \text{ psi} < f_c^{'} < 8000 \text{ psi}$$

A closed-form solution can be derived to determine x but it is simpler to use an iterative process where x is guessed and equilibrium is checked until the solution falls within a specified tolerance. An algorithm to do this is:

1. Guess x
2. $a = \beta_1 x$
3. $\varepsilon_s^{'} = \dfrac{0.003(x - d')}{x}$
4. f'_s is determined from ε'_s as defined the material properties.
5. $\varepsilon_s = \dfrac{0.003(d - x)}{x}$

6. f_s is determined from ε_s as defined the material properties.
7. $T = A_s f_s$
8. $C_s = A'_s f'_s$
9. $C_c = 0.85 f'_c ab$
10. $C = C_c + C_s$
11. Does $C = T$? If yes, stop - x is ok, if not revise x and go to step 2.

Figure 7 shows the cross-section and corresponding strain and stress profiles for a beam at the ultimate load state.

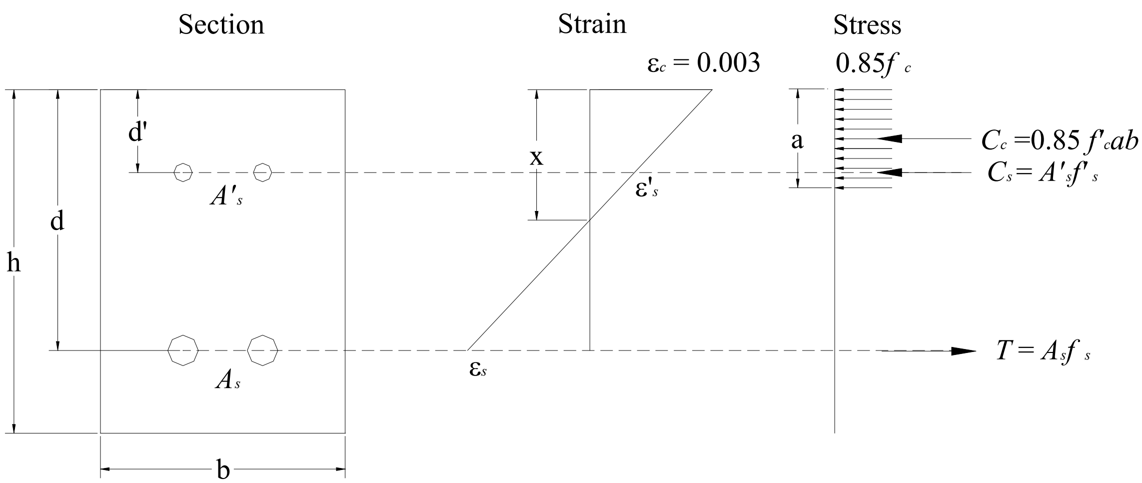


Figure 7 - Strain-stress plot at ultimate load

To determine the ultimate (nominal) moment capacity, M_n, use the definition of a moment – force multiplied by a moment arm. Choose either compression or tension forces and multiply them by their respective distances from the neutral axis. If compression forces are used, M_n is:

$$M_n = 0.85 f_c^{'} ab\left(d - \frac{a}{2}\right) + A_s^{'} f_s^{'}\left(d - d'\right)$$

This assumes that the compression steel is indeed in compression. If the steel were in tension (if $x < d'$) then the second term in the equation above would be negative. The ultimate curvature is equal to:

$$\Phi_{ult} = \frac{0.003}{x}$$

The moment capacity of a beam at ultimate load should be higher than at yielding. This is due to the requirements set forth by ACI that a beam must fail by steel yielding before concrete crushing. However, the experimental specimens may not follow this rule if the moment capacity at ultimate is lower than at yield an adjustment to the yield point should be made. To do this, first extend a line from the origin of the M-Φ diagram to the yield point (1). Next, draw a horizontal line to the left from the ultimate point until it intersects with the first line. This intersection is the adjusted yield point (see figure 8). It indicates that the concrete crushes before the tension yields.

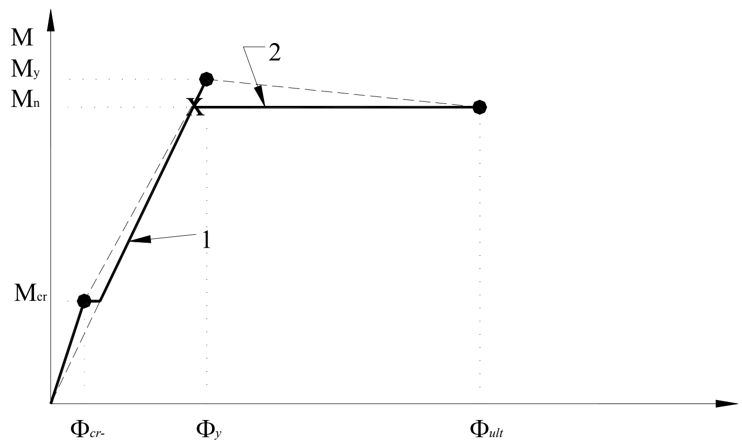


Figure 8 - Adjusted yield point on the M-Φ diagram

The ultimate displacement is comprised of two parts – elastic displacement and inelastic displacement. Elastic displacement includes all displacement that occurs prior to yielding and is equivalent to Δ_y, discussed earlier. Inelastic displacement is comprised of displacement that occurs after yielding has occurred. To quantify inelastic displacement plastic hinge analysis is used. A plastic hinge is a fully hinged portion of a beam that is free to rotate and is able to maintain moment-carrying capacity. A plastic hinge is modeled in a curvature diagram as a rectangle of height h_p,equal to $\Phi_{ult} - \Phi_y$, and length L_p where $h/2 < L_p < h$. Plastic hinge analysis is a modified moment area method. Two relationships are important to the method and are illustrated in figure 9. These are:

- The change in slope (θ) between the tangents of the deflected shape at two points is equal to the area under the *M/EI* (or Φ) diagram between the same two points.

- The deflection of a beam at one point with respect to the tangent at another point due to θ is equal to the distance between the points multiplied by θ.

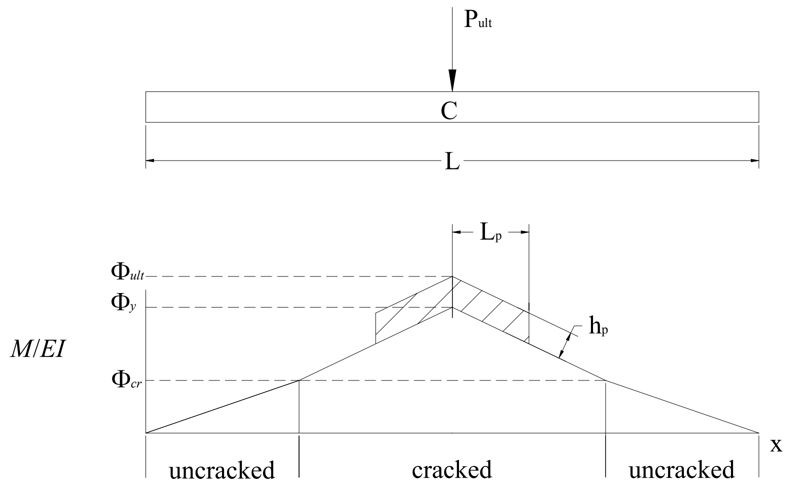


Figure 9 - Plastic Hinge Analysis

To calculate the inelastic displacement, first calculate θ_p:

$$\theta_p = L_p h_p$$

where:

$$\frac{h}{2} < L_p < h$$

$$h_p = \Phi_{ult} - \Phi_y$$

Then calculate δ_p as: $$\delta_p = \theta_p \left(\frac{L}{2} - \frac{L_p}{2} \right)$$

The total displacement is equal to $\delta_p + \delta_y$. To determine the ultimate load (P_{ult}) follow the same procedure as in finding P_y and P_{cr}.

$\varepsilon_c > 0.003$

Even though the beam has reached its nominal capacity as defined by ACI 318-99, the beam can still resist moment when subjected to increasing delections. Moment capacity will of course decrease and deflections will increase as the beam undergoes higher levels of curvature until the beam finally collapses.

To determine post-ultimate moment capacity, a strain level at the outer compressive surface must first be established. Strains of 0.005, 0.007 and 0.01 are arbitrarily selected to determine behavior for $\varepsilon_c > 0.003$. The procedure to determine the moment capacity and other quantities at a post-ultimate strain level is essentially the same as at the ultimate strain level with a couple of changes. First, concrete stress must be modeled with the Hognestad equation, not the Whitney stress block. The Whitney stress block is only valid for a strain of 0.003. A numerical approximation will be used to determine the area of the stress block since it is not a geometrically convenient shape. To do this, the distance x from the outer compression surface to the neutral axis is guessed. The stress block is then sliced into n pieces of equal thickness and each stress level is multiplied by the rectangular area of each slice (see figure 10).

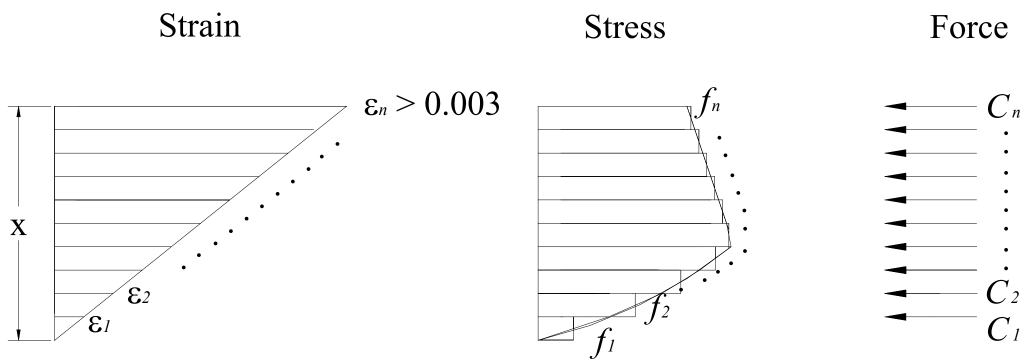


Figure 10 - Stress-strain relation for ε_i>0.003 using the Hognestad model

Equilibrium is checked and the process is repeated until the solution falls within a specified tolerance. An algorithm to do this is:

1. Guess x
2. Slice stress block into n pieces of depth x/n
3. Calculate the strain at each interval as $\varepsilon_i = \dfrac{n\varepsilon_n}{i}$
4. Calculate the corresponding stress, f_s at each interval using the Hognestad model as defined on page 2.
5. Calculate the force produced by each slice as: $C_i = \left(\dfrac{f_{c(i-1)} + f_{ci}}{2}\right)$
6. $C_c = \sum_{i=1}^{n} C_i$
7. Calculate the strain in the top steel as $\varepsilon_s' = \dfrac{x\varepsilon_n}{x-d'}$
8. f'_s is determined from ε'_s as defined by the model on page 1.
9. Calculate the strain in the bottom steel as $\varepsilon_s = \dfrac{x\varepsilon_n}{d-x}$
10. f_s is determined from ε_s as defined by the model on page 1.
11. $T = A_s f_s$

12. $C_s = A'_s f'_s$
13. $C = C_c + C_s$
14. Does $C = T$? If yes, stop - x is ok, if not revise x and go to step 2.

Figure 11 shows the cross-section and corresponding strain and stress profiles for a beam at a post-ultimate load state.

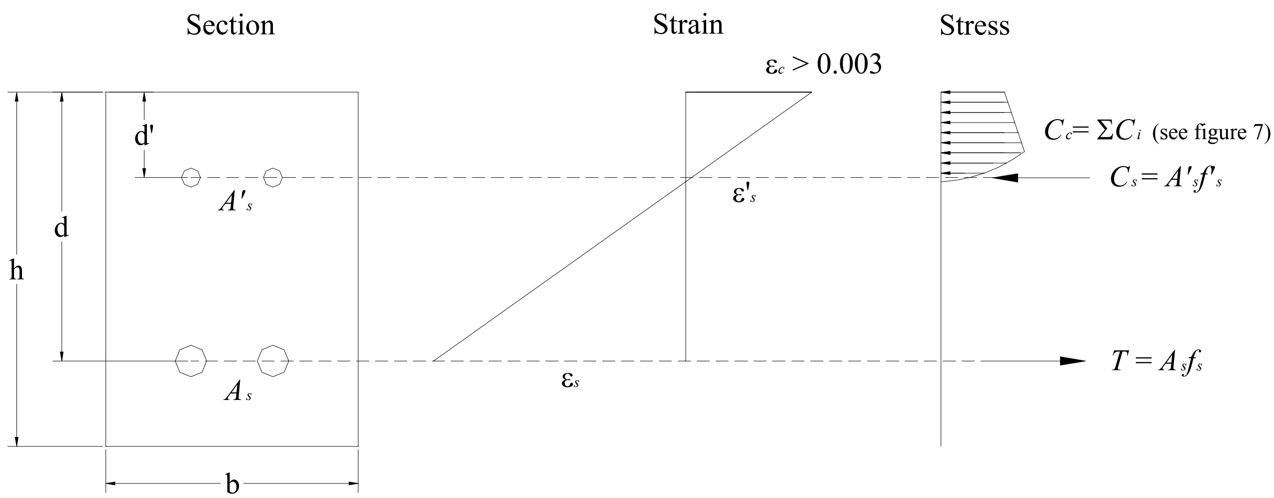


Figure 11 - Strain-stress plot at post-ultimate load

To determine the ultimate (nominal) moment capacity, $M_{\varepsilon n}$, multiply each compressive force by its respective moment arm. The moment arm, L_{arm}, for each slice in the approximated stress curve is equal to:

$$L_{arm_i} = d - x + (i-1)t + \frac{t}{2}$$

$M_{\varepsilon n}$ is equal to:

$$M_{\varepsilon n} = \sum_{i=1}^{n} C_i L_{arm_i} + A_s' f_s'$$

This assumes that the compression steel is indeed in compression. If the steel were in tension (if $x < d'$) then the second term in the equation above would be negative. The post-ultimate curvature is equal to:

$$\Phi_{\varepsilon_n} = \frac{\varepsilon_n}{x}$$

The post-ultimate deflection and load is determined using the same procedure as at the ultimate state.

Shear Strength of Beam Specimens

The shear strength provided by concrete in the beam specimen is to be calculated using the equations as defined in ACI 318-99 Section 11.3. Either equations 11-3 or 11-5 may be used.

$$V_c = 2\sqrt{f_c} b_w d \tag{11-3}$$

$$V_c = \left(1.9\sqrt{f_c} + \frac{2500\rho_w V_u d}{M_u}\right) b_w d \tag{11-5}$$

The shear strength provided by transverse steel in the beam specimen is calculated by:

$$V_s = \frac{A_v f_y d}{s}$$

The total shear strength (V_n) is equal to the sum of V_c and V_s. If the beam experiences a shear of V_n before failing in flexure, the beam is said to fail in shear. However, if the shear force associated with M_n is lower than V_n, a flexural failure is reached. Physically, these different failure modes can be seen in the crack patterns that occur on the specimen during loading. A shear failure will appear as cracks spanning across the beam web, between the compressive and tensile steel (fig 12). A flexural failure will exhibit cracks normal to the tensile steel near the application of the load, fanning outward further from the applied load (fig 13).

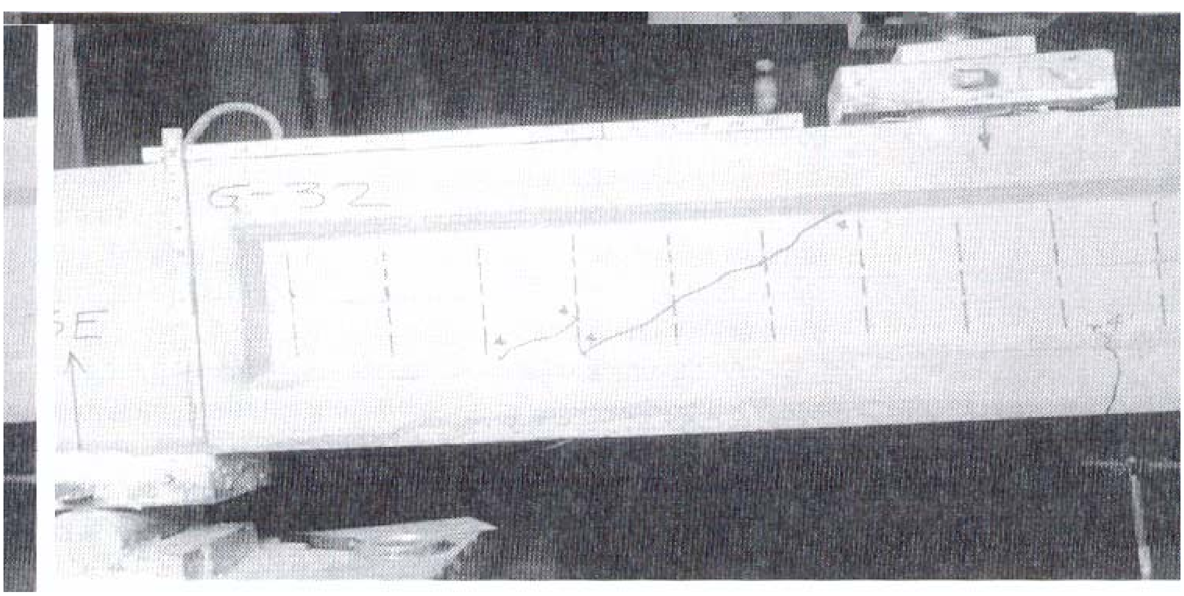


Figure 12– Web-shear cracks

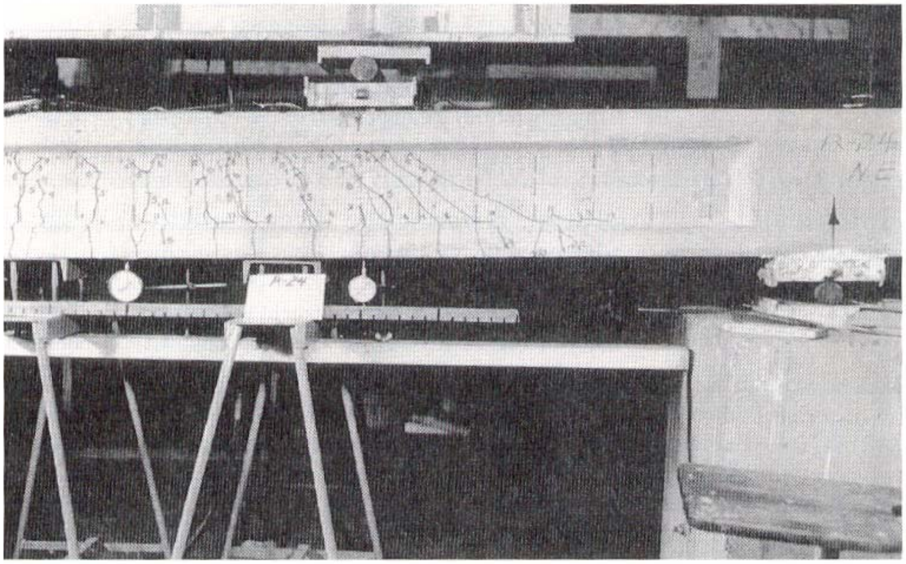


Figure 13- Flexure-shear cracks

Appendix

Experimental Beam Dimensions

Beam dimensions for all four laboratory specimens are shown in figure 14. Dimensions should be double-checked prior to testing to ensure accuracy of results.

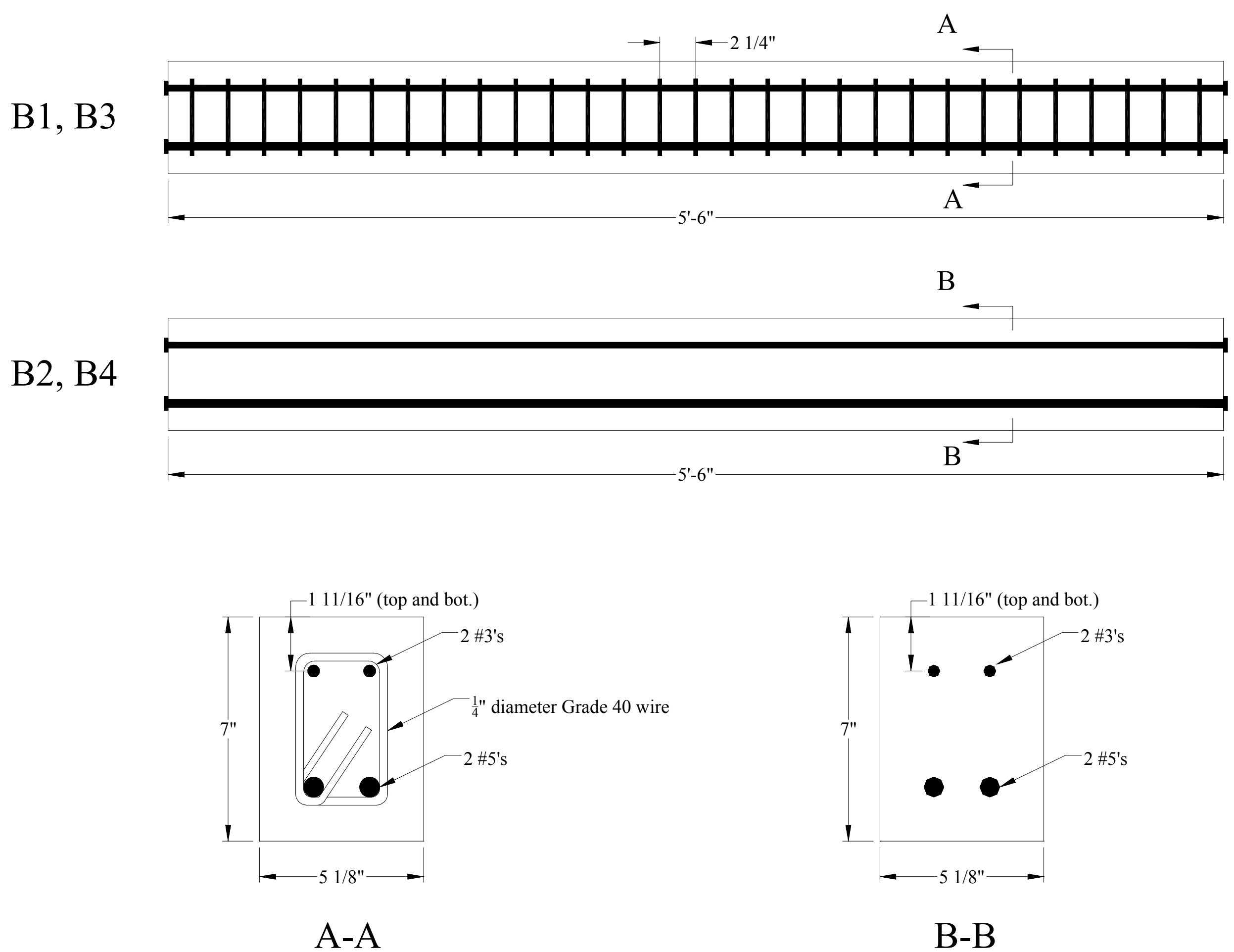


Figure 14 - Experimental beam dimensions

Example Problem

Geometry

b	5.13	in
h	7.00	in
d	5.31	in
d'	1.69	in
n	8.60	E_s/E_c
A_s	0.62	in^2
A'_s	0.22	in^2
L	66.00	in

Materials

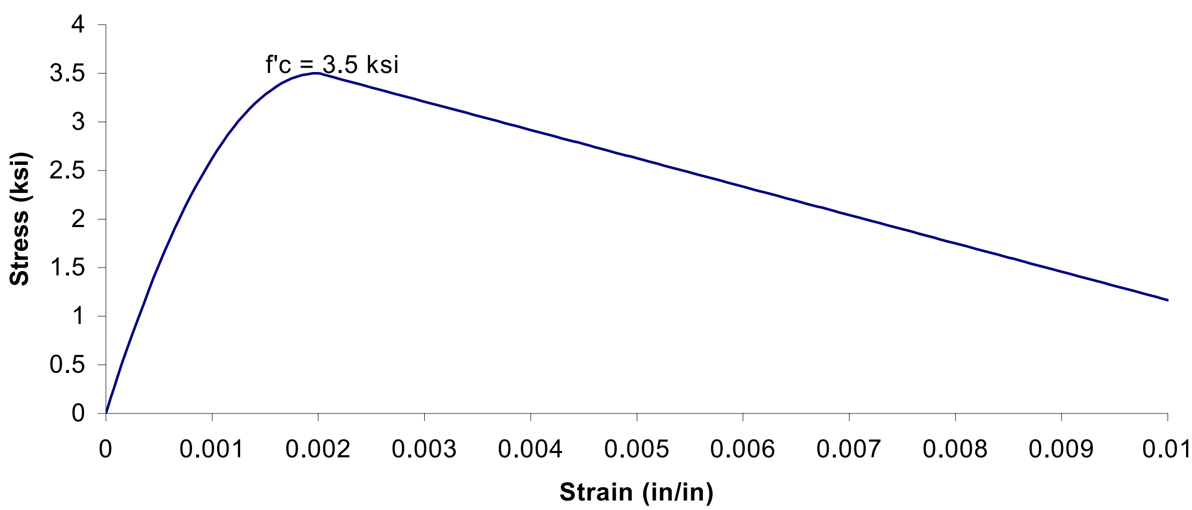
Concrete

f'_c	3.5	ksi
E_c	3372.165	ksi
ε_o	0.002	in/in
ε_{85}	0.0038	in/in
f_{85}	2.975	ksi

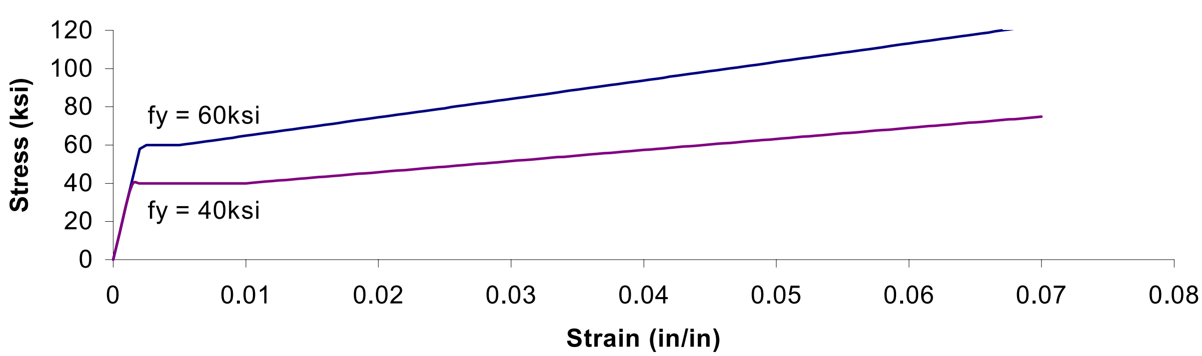
Steel

f_y	60	ksi
f'_y	40	ksi
E_s	29000	ksi
E_{sh}	967	ksi
E'_{sh}	580	ksi
ε_{sh}	0.0050	in/in
ε'_{sh}	0.0100	in/in
ε_y	0.0021	in/in
ε'_y	0.0014	in/in

Hognestad Concrete Model



Steel Model



Flexural Behavior

Prior to Cracking

I_g	146.49	in^4
f_r	0.44	ksi
c	3.50	in
M_{cr}	18.57	in-kips
Φ_{cr-}	3.76E-05	/in

Just After Cracking

Φ_{cr+}	7.90E-05	/in

Yielding

a	1.00	
b	2.73	
c	-12.15	
y'	2.38	in
I_{cr}	69.68	in^4
f_c	5.65	ksi
M_y	165.68	in-kips
Φ_y	7.05E-04	/in

Ultimate ε_c = 0.003

a	2.06	in
x	2.42	in
e'_s	0.0009	in/in
f'_s	26.39	ksi
e_s	0.0036	in/in
f_s	60.00	ksi
T	37.20	kips
C_s	5.81	kips
C_c	31.39	kips
C	37.20	kips
$T\text{-}C$	0.0006	kips
M_{ult}	155.50	in-kips
Φ_{ult}	1.24E-03	/in

ε_c = 0.005

x	2.192048	in
n	100	
dx	0.02192	in
A_{slice}	0.112342	in^2
ε_n	0.005	in/in
ε'_s	0.001151	in/in
f'_s	33.37495	ksi
ε_s	0.007118	in/in
f_s	62.04707	ksi
T	38.46919	kips
Cs	7.342489	kips
Cc	31.12659	kips
C	38.46908	k
$T\text{-}C$	0.000106	k
M_{005}	160.5191	in-kips
Φ_{005}	0.002281	/in

ε_c = 0.010

x	2.634185	in
n	100	
dx	0.026342	in
A_{slice}	0.135002	in^2
ε_n	0.01	in/in
ε'_s	0.003594	in/in
f'_s	40	ksi
ε_s	0.010168	in/in
f_s	64.99528	ksi
T	40.29707	kips
Cs	8.8	kips
Cc	31.49652	kips
C	40.29652	k
$T\text{-}C$	0.000548	k
M_{010}	153.7346	in-kips
Φ_{010}	0.003796	/in

ε_c = 0.007

x	2.283891	in
n	100	
dx	0.022839	in
A_{slice}	0.117049	in^2
ε_n	0.007	in/in
ε'_s	0.001828	in/in
f'_s	40	ksi
ε_s	0.009283	in/in
f_s	64.13977	ksi
T	39.76665	kips
Cs	8.8	kips
Cc	30.96665	kips
C	39.76665	k
$T\text{-}C$	9.59E-06	k
M_{007}	161.2221	in-kips
Φ_{007}	0.003065	/in

Load - Displacement

L	66

Cracking

Φ_{cr-}	3.76E-05	/in
θ_{cr}	0.00062	rad
δ_{cr}	0.013647	in
P_{cr}	1.125512	kips

Yield

Φ_{cr-}	3.76E-05	/in
Φ_{y}	0.000705	/in
L_{uncr}	3.698932	in
L_{cr}	29.30107	in
θ_1	6.95E-05	rad
θ_2	0.001102	rad
θ_3	0.019558	rad
δ_1	0.000171	in
δ_2	0.020213	in
δ_3	0.454382	in
δ_{cr}	0.474766	in
P_{cr}	9.424532	kips

Ultimate

Φ_{ult}	0.001238	/in
$L_p/2$	5.25	in
h_p	0.000533	in
θ_{pl}	0.0028	rad
δ_{pl}	0.085062	in
δ_e	0.474766	in
δ_{ult}	0.559828	in
P_{ult}	9.424532	kips

e = 0.007

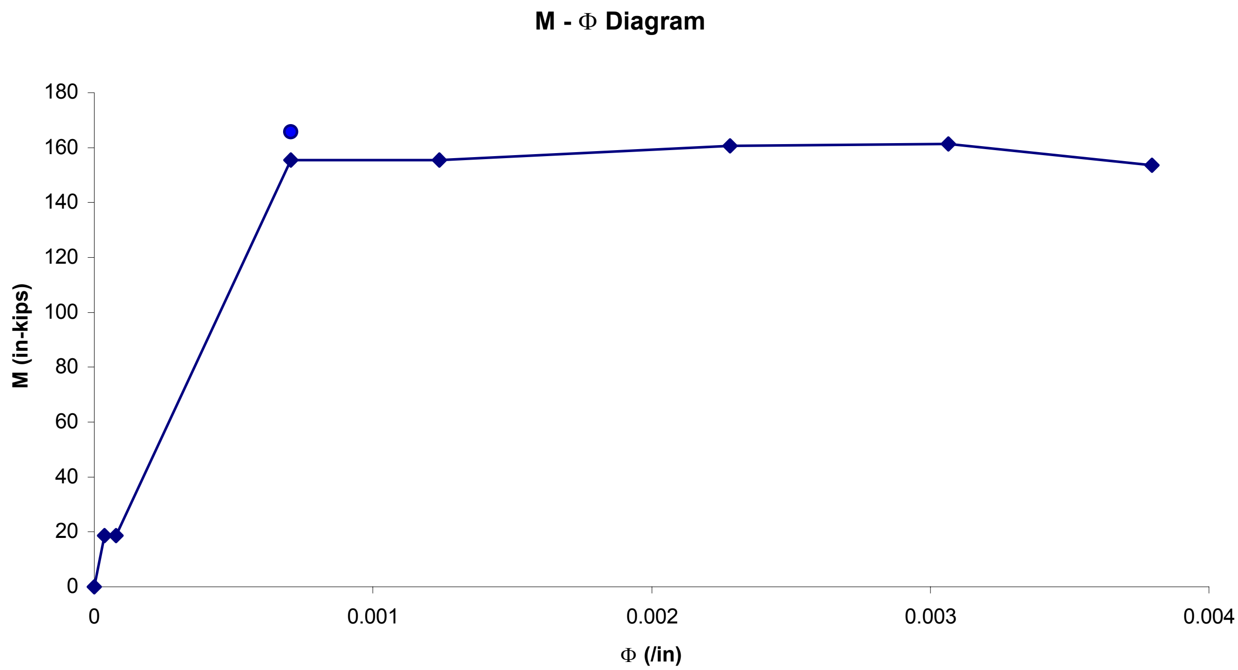
Φ_{007}	0.003065	/in
$L_p/2$	5.25	in
h_p	0.00236	in
θ_{pl}	0.012389	rad
δ_{pl}	0.376327	in
δ_e	0.474766	in
δ_{007}	0.851093	in
P_{007}	9.771036	kips

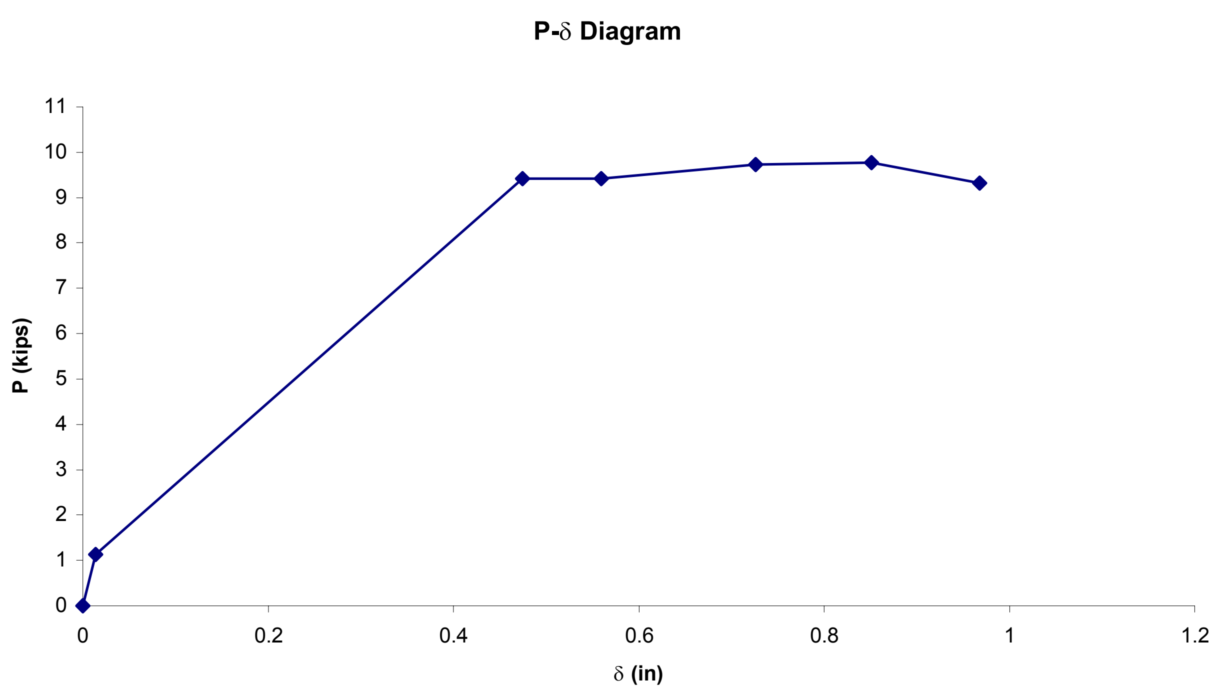
e = 0.005

Φ_{005}	0.002281	/in
$L_p/2$	5.25	in
h_p	0.001576	in
θ_{pl}	0.008274	rad
δ_{pl}	0.251308	in
δ_e	0.474766	in
δ_{005}	0.726074	in
P_{005}	9.728432	kips

e=0.01

Φ_{010}	0.003796	/in
$L_p/2$	5.25	in
h_p	0.003091	in
θ_{pl}	0.016229	rad
δ_{pl}	0.492946	in
δ_e	0.474766	in
δ_{010}	0.967712	in
P_{010}	9.317251	kips


M - Φ Diagram
M (in-kips)
0
20
40
60
80
100
120
140
160
180
0
0.001
0.002
0.003
0.004
Φ (/in)


P-δ Diagram
P (kips)
0
1
2
3
4
5
6
7
8
9
10
11
0
0.2
0.4
0.6
0.8
1
1.2
δ (in)