Homework 7

Due: Thursday 6/1/2017.

Reading assignment: Chapter 18 in the textbook. Chapter 2 (Cholesky factorization), sections 4.2 and 4.3 (Newton’s method for nonlinear equations), and chapter 5 (Unconstrained minimization) in the EE133A Lecture Notes.

Homework problems

1. Exercise A11.1 (a,b,c).
2. Exercise A11.8 (b,c,d).
5. Exercise A14.8. You can follow algorithm 18.3 on page 372 of the textbook (Levenberg-Marquardt method), or the simpler basic Gauss-Newton method (algorithm 18.1) on page 367. In algorithm 18.3 you can take $\lambda^{(1)} = 1$ as starting value.
6. The figure shows a two-link robot manipulator in a plane.

![Two-link robot manipulator figure]

The robot manipulator endpoint is at the position

$$p = L_1 \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} + L_2 \begin{bmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{bmatrix},$$

where $L_1$ and $L_2$ are the lengths of the first and second links, $\theta_1$ is the first joint angle, and $\theta_2$ is second joint angle. We assume that $L_2 < L_1$, i.e., the second link is shorter than the first. We are given a desired endpoint position $p^{\text{des}}$, and seek joint angles $\theta = (\theta_1, \theta_2)$ for which $p = p^{\text{des}}$. In this exercise you will use the Levenberg-Marquardt algorithm to find joint angles, by minimizing $\|p - p^{\text{des}}\|^2$. 
(a) Identify the function \( f(\theta) \) in the nonlinear least squares problem, and give its derivative \( Df(\theta) \).

(b) Implement the Levenberg-Marquardt algorithm (Algorithm 18.3 in the textbook) to solve the nonlinear least squares problem. Try your implementation on a robot with \( L_1 = 2 \), \( L_2 = 1 \), and the desired endpoints

\[
p^{\text{des}} = (1.0, 0.5), \quad p^{\text{des}} = (-2.0, 1.0), \quad p^{\text{des}} = (-0.2, 3.1).
\]

Start the iteration with \( \theta^{(1)} = 0 \) and \( \lambda^{(1)} = 1 \). Plot the residual norm \( \| f(\theta^{(k)}) \| \) versus iteration number \( k \).

Note that the norm of the third desired endpoint \((-0.2, 3.1)\) exceeds \( L_1 + L_2 = 3 \), so there are no joint angles for which \( p = p^{\text{des}} \). Explain the angles your algorithm finds in this case.

**Remark.** Although it does not matter for the exercise, this problem can in fact be solved analytically. The expression for the endpoint \( p \) can be written as

\[
p = \begin{bmatrix} L_1 + L_2 \cos \theta_2 \\ L_2 \sin \theta_2 \end{bmatrix} \begin{bmatrix} -L_2 \sin \theta_2 \\ L_1 + L_2 \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix}.
\]

We can note that

\[
\| p \|^2 = (L_1 + L_2 \cos \theta_2)^2 + (L_2 \sin \theta_2)^2
\]

\[
= L_1^2 + L_2^2 + 2L_1L_2 \cos \theta_2.
\]

If we are given \( p = p^{\text{des}} \) we can solve this equation for \( \theta_2 \). When \( L_1 - L_2 < \| p^{\text{des}} \| < L_1 + L_2 \), there are two choices of \( \theta_2 \) (one positive and one negative). For each solution \( \theta_2 \) we can then find \( \theta_1 \) from (1).