Additional problems for homework #2

1. **Partial addition and intersection.** The intersection and sum

\[ C_1 \cap C_2 = \{ x \mid x \in C_1, x \in C_2 \}, \quad C_1 + C_2 = \{ y_1 + y_2 \mid y_1 \in C_1, y_2 \in C_2 \} \]

of two convex sets \( C_1 \) and \( C_2 \) are convex sets. The following is an extension of these two operations. Suppose \( C_1 \) and \( C_2 \) are convex sets in \( \mathbb{R}^{m+n} \). Define

\[ C = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in C_1, (x, y_2) \in C_2 \}. \]

Show that \( C \) is convex.

2. **Conic representation of convex set.** Let \( C \) be a nonempty convex set in \( \mathbb{R}^n \). Define

\[ K = \{(x, t) \in \mathbb{R}^{n+1} \mid t \geq 0, x \in tC \}. \]

(For a scalar \( t \), the set \( tC \) is defined as \( tC = \{ty \mid y \in C\} \).) Show that \( K \) is a convex cone.

An example of this construction, with \( C \) a unit norm ball, is the norm cone \( C = \{(x, t) \mid t \geq 0, \|x\| \leq t \} \) of the lecture notes, page 2-8.

3. **Polar of a convex set.** The polar of a set \( C \subseteq \mathbb{R}^n \) is defined as

\[ C^\circ = \{ y \in \mathbb{R}^n \mid y^T x \leq 1 \text{ for all } x \in C \}. \]

(a) Show that \( C^\circ \) is convex (even if \( C \) is not).

(b) Suppose \( C \) is a cone. How are the polar \( C^\circ \) and the dual cone \( C^* \) related?

(c) What are the polars of the following sets?

\[ C_1 = \{ x \mid \|x\|_2 \leq 1 \}, \quad C_2 = \{ x \mid \|x\|_1 \leq 1 \}, \quad C_3 = \{ x \mid 1^T x = 1, x \succeq 0 \}. \]

(Here \( 1 \) denotes the vector of ones.)

(d) By definition of \( C^\circ \), we have

\[ y^T x \leq 1 \text{ for all } x \in C, y \in C^\circ. \]

(For the set \( C_1 \) this is the Cauchy-Schwarz inequality.) Show that this implies that \( C \subseteq (C^\circ)^\circ. \)
(e) Suppose $C$ is closed, convex, and $0 \in \text{int } C$. Show that

$$(C^\circ)^\circ = C.$$ 

Given the result of part (d), it is sufficient to show that $(C^\circ)^\circ \subseteq C$. In other words, show that if $x \notin C$ then $x \notin (C^\circ)^\circ$.

**Hints.** Use the strict separating hyperplane theorem of page 49 of the textbook. If $C$ is a closed convex set and $x \notin C$ then there exists a vector $a \neq 0$ and a scalar $b$ such that

$$a^T x > b, \quad a^T z \leq b \text{ for all } z \in C.$$ 

Then use the fact that $0 \in \text{int } C$ to argue that one can take $b = 1$. (The condition $0 \in \text{int } C$ means that there exists a small ball $B = \{u \mid \|u\|_2 \leq \epsilon\}$ with $\epsilon > 0$ and $B \subseteq C$.)