CEE 142L

Reinforced Concrete Structures
Laboratory

Slab Experiment
Analysis of Two-Way Slabs

The predicted load-displacement history of a two-way slab is simplified as an ideal function with three distinct regions as shown in Figure 1.

Region ‘I’ is the linear elastic uncracked region where the slab is very stiff, that is, the load-deflection curve is steep. Behavior in this region is predicted by elastic theory.

Region ‘II’ begins after cracking and is also idealized as linear. Cracks start to form normal to line of principal moment (tension face). The slope of the load-deflection curve is less steep since the section now has a reduced stiffness due to cracking. During this phase, steel reinforcement loses bonding strength and cracks begin to form in the slab. As the slab reaches and exceeds yielding, the crack widths will increase. Behavior in this region is predicted by a modified elastic theory.

To account for cracking, modified Kirchhoff plate theory is used, i.e, plate rigidity (D) is reduced to account for cracking.

\[ D_{cr} = \frac{D \cdot I_{cr}}{I_{gross}} \]

Determination of I_{cr} is subjective because cracks and reinforcement are not usually parallel or orthogonal. Also, crack orientation changes with location. It is common to use I/unit width. Thus I_{cr} and I_{gross} have units of length\(^3\). I_{cr} is calculated based on geometry and materials. It is convenient to use unit width equal to spacing of reinforcement.

M_{cr} and M_y is calculated using identical methods as a beam with unit width.
In region ‘III’ the reinforcement is yielding and the load-deflection curve is constant. This indicates that the slab has reached its maximum load carrying capacity. Behavior in this region is predicted by yield-line analysis.

![Graph showing load-deflection relationship with regions I, II, and III]

**Figure 1**

**Elastic Theory and Analysis**

**Equilibrium**

To begin the derivation of equations necessary to predict the behavior of a two-way slab, consider the equilibrium of a uniformly loaded beam section. The change in shear over the length \( dx \) is calculated as:

\[
\sum F = 0 \Rightarrow \frac{\partial V}{\partial x} \cdot dx = -w \cdot dx
\]

\[
\frac{\partial V}{\partial x} = -w
\]

(1)

The change in moment over the length \( dx \) is calculated as:
\[ \sum M_x = 0 \Rightarrow \frac{-\partial M}{\partial x} \ dx + \left( V + \frac{\partial V}{\partial x} \ dx \right) + \frac{w \cdot dx^2}{2} \]
\[ \Rightarrow \frac{-\partial M}{\partial x} \ dx + dx(V + 0) \]
\[ V = \frac{\partial M}{\partial x} \Rightarrow \frac{\partial^2 M}{\partial x^2} = -w \]

**Figure 2**

Now consider the equilibrium of the two-dimensional slab element in Figure 2 and 3. Figure 2 describes the equilibrium of shear forces. Summing these, an expression relating shear and loading is obtained:

\[ \frac{\partial V}{\partial x} \ dx \ dx \ dy + \frac{\partial V}{\partial y} \ dx \ dy = -w \ dx \ dy \]
\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = -w \]

Figure 3 shows the moment equilibrium of the slab element. By taking the sum of the moments equal to zero, the expressions for shear are:
Combining equations 2, 3 and 4 derives the equation of statics for a uniformly loaded two-dimensional element:

$$\frac{\partial M_x^2}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial M_y^2}{\partial y^2} = -w$$

(5)

where plate rigidity (D) is given as:

$$D = \frac{Et^3}{12(1-\nu^2)}$$

Figure 3

Deflections

Substituting the equations for moments into equation 5 gives:

$$\frac{\partial^4 z}{\partial x^4} + 2 \frac{\partial^4 z}{\partial x^2 \partial x^2} + \frac{\partial^4 z}{\partial y^4} = -\frac{u}{D}$$

(6)

where plate rigidity (D) is given as:

$$D = \frac{Et^3}{12(1-\nu^2)}$$

here E is the modulus of elasticity and \(\nu\) is poisson’s ratio (0.15-0.20 for concrete)
Moment-Deformation Relationships

The moment-deformation relationships for a uniformly loaded two-way slab are based on two assumptions:

- The slab material is isotropic and obeys Hooke’s law.
- Deflections are small compared to the slab thickness.

For linear elastic material, Hooke’s law is given as:

\[
\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)
\]
\[
\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)
\]
\[
\tau_{xy} = \frac{\gamma_{xy}E}{2(1+\nu)}
\]

The curvature and strains of a slab in a given cross-section are the same as in a beam, that is:

\[
\phi_x = \frac{\varepsilon_x}{z} = \frac{\partial^2 u}{\partial x^2}
\]

\[
\therefore \varepsilon_x = -z\frac{\partial^2 u}{\partial x^2}, \quad \varepsilon_y = -z\frac{\partial^2 u}{\partial y^2}
\]

Combining this and Hooke’s law:

\[
\sigma_x = -z\frac{E}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}\right)
\]

The moment per unit width is:
\[ M_x = \int_{-l}^{l} \sigma_x z dz = \frac{E l^3}{1 - \nu^2} \left( \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} \right) \]

\[ M_x = -D \left( \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} \right) \]

\[ M_y = -D \left( \frac{\partial^2 u}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x^2} \right) \]

\[ M_{xy} = -\frac{\partial^2 u}{\partial \xi \partial \eta} D(1 - \nu) \]

Boundary Conditions

The slab specimens to be tested are all simply supported. Support types define the mathematical boundary conditions used to solve the differential equations for moment and deflection. Navier and Levy solutions are utilized to solve these differential equations.

Navier solution:

\[
u = \frac{16w}{\pi^3 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \]

\[
\frac{m^2}{a^2 + \frac{n^2}{b^2}} \]

\[ m = 1, 3, 5... \]

\[ n = 1, 3, 5... \]

Levy Solution (more rapid convergence):

\[
u = \frac{4w a^4}{\pi^3 D} \sum_{m=1, 3, 5,..}^{\infty} \frac{1}{m^3} \left[ 1 - \frac{\alpha_m}{2} \frac{\tanh \alpha_m + 2 \cosh \alpha_m y}{\cosh \alpha_m b} + \frac{\alpha_m y}{b \cosh \alpha_m b} \sinh \frac{2\alpha_m y}{b} \right] \frac{m \pi x}{a} \]
where
\[
\alpha_m = \frac{m \cdot \pi \cdot b}{2 \cdot a}
\]

In order to calculate slab deflections using Levy solution, uniformly distributed load (w (psi)) must be calculated. By solving w using bending moment equations given below and utilizing obtained w in Levy solution, moment-deflection relationship can be calculated for selected moments. Coordinate system used for given equations is given below.

\[
M_{xx} = \frac{w \cdot x \cdot (a-x)}{2} + (1-\nu) \cdot w \cdot a^2 \cdot \pi^2 \cdot \sum_{m=1,3,5,...} \frac{m^2}{a} \cdot [A_m \cdot \cosh \frac{m \cdot \pi \cdot x \cdot y}{a} + B_m \cdot \left(\frac{m \cdot \pi \cdot y}{a}\right) \cdot \sinh \frac{m \cdot \pi \cdot y}{a}]
\]

\[
M_{yy} = \frac{\nu \cdot w \cdot x \cdot (a-x)}{2} - (1-\nu) \cdot w \cdot a^2 \cdot \pi^2 \cdot \sum_{m=1,3,5,...} \frac{m^2}{a} \cdot [A_m \cdot \cosh \frac{m \cdot \pi \cdot x \cdot y}{a} + B_m \cdot \left(\frac{m \cdot \pi \cdot y}{a}\right) \cdot \sinh \frac{m \cdot \pi \cdot y}{a}]
\]
where

\[ A_m = -2 \cdot \left( \alpha_m \cdot \tanh \alpha_m + 2 \right) / \left( \pi^5 \cdot m^5 \cdot \cosh \alpha_m \right) \]

\[ B_m = \frac{2}{\pi^5 \cdot m^5 \cdot \cosh \alpha_m} \]

Along the x-axis

\[ M_{xx}(y = 0) = \frac{w \cdot x \cdot (a - x)}{2} - w \cdot a^2 \cdot \pi^2 \sum_{m=1,3,5,...}^{\infty} m^2 \cdot \left[ 2 \cdot v \cdot B_m - (1 - v) \cdot A_m \right] \cdot \sin \frac{m \cdot \pi \cdot x}{a} \]

\[ M_{yy}(y = 0) = \frac{v \cdot w \cdot x \cdot (a - x)}{2} - w \cdot a^2 \cdot \pi^2 \sum_{m=1,3,5,...}^{\infty} m^2 \cdot \left[ 2 \cdot B_m - (1 - v) \cdot A_m \right] \cdot \sin \frac{m \cdot \pi \cdot x}{a} \]

Maximum values are at midspan, where \( x = a/2 \) and \( y = 0 \), and \( M_{xx} = M_{yy} \) for a square plate.