PROPAGATION AND EXCITATION OF PLASMA DENSITY-GRADIENT WAVES ACROSS A MAGNETIC FIELD.

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ABSTRACT.

A general dispersion relation for low-frequency, electrostatic oscillations of an inhomogeneous low-density, fully-ionized plasma in a strong magnetic field B is derived from the macroscopic equations for waves which propagate at a slight angle θ to the direction of the density-gradient drift. Two sets of waves are found in the limits where the electrons are assumed to move freely along B or not at all along B. In the latter case, one wave travels at the macroscopic drift velocity; a physical description of this phenomenon is given in terms of the microscopic particle motions. The transition between the two sets of waves occurs for small values of θ of the order of $(m/M)^{1/2}$ or $(ne\eta/B)^{1/2}$, where η is the resistivity. In this region of θ, long-wavelength ion cyclotron waves and density-gradient waves may be excited by the pressure gradient, even if no longitudinal drift exists; the growth rate, computed numerically, is of the same order of magnitude as or smaller than the real part of the frequency for conditions which exist in a thermally-ionized plasma or in a stellarator.

RECEIVED
JUN 28 1963
OFFICE OF THE EDITOR
THE PHYSICS OF FLUIDS
Acknowledged JUL 1 1963

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PP-1719

I - INTRODUCTION.

Low-frequency oscillations in an inhomogeneous plasma created by thermal ionization have recently been observed by D'Angelo and Motley[1]. These oscillations seem to propagate across the magnetic field with the same velocity as the pressure-gradient drift; that this should follow from the macroscopic equations of the plasma has been noted by D'Angelo[2]. However, that a density perturbation should propagate with the macroscopic drift seems at first to contradict the microscopic picture that, in the absence of collisions, the ions are tied to the lines of force, which are stationary. We reexamine this problem, starting with the basic equations and approximations given in Sec. II, and find in Sec. III that in the limit of strictly perpendicular propagation these drift waves are a physically real phonomenon.

We then inquire whether or not the velocity of this wave would be greatly affected by a small component of $\underset{\sim}{k}$, the propagation vector, along the magnetic field $\underset{\sim}{B}$. It is known, for example, that electrostatic electron oscillations at the lower hybrid frequency exist only in theory, for if the angle of propagation deviates from $\pi/2$ by as little as the square root of the electron-ion mass ratio, this frequency no longer exists. We find that the same is true of the density gradient waves; for angles large compared to the root of the mass ratio, Sec. IV

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[1] N. D'Angelo and R. W. Motley - Phys. Fluids - 6, 422 (1963).

[2] N. D'Angelo - Phys. Fluids - 6, 592 (1963).

shows that the waves become the ion cyclotron waves, propagating perpendicular to $\underset{\sim}{B}$, discussed by Motley and D'Angelo[3] and by Drummond and Rosenbluth[4]. Our treatment differs from that of Ref. 3 only in the inclusion of a density gradient correction.

To effect the transition between perpendicular propagation and propagation at sizable angles, one must take into account the finite velocity of the electrons in their motion along $\underset{\sim}{B}$. In Sec.V we treat the case in which this velocity is limited by electron inertia; in Sec.VI, the case in which it is limited by resistivity. We find that in the region of small but finite $k_{/\!/}$ the ion cyclotron waves not only suffer a change in frequency but may become unstable and grow at the expense of the density gradient. The relevance of this effect to anomalous diffusion is briefly discussed in Sec.VII.

These oscillations may also be studied from the point of view of the Vlasov equation, as has been done elegantly by Rosenbluth[4,5] and his collaborators. We feel, however, that in thermally ionized plasmas, where collisions are frequent, the macroscopic equations are likely to be more accurate. In any case, the effects of small but finite $k_{/\!/}$ are more easily brought into focus from our point of view. We have not introduced any zero-order drift along $\underset{\sim}{B}$, since Landau damping would be expected to have a serious effect on the excitation of the waves in this case, and the Boltzmann equation must then be used.

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[3] R.W. Motley and N. D'Angelo - Phys. Fluids - 6, 296 (1963).

[4] W. Drummond and M. Rosenbluth - Phys. Fluids - 5, 1507 (1962)

[5] N. Krall and M. Rosenbluth - Phys. Fluids - 6, 254 (1963).

II - FUNDAMENTAL EQUATIONS.

For a fully-ionized plasma, the first two moments of the Boltzmann equation give, for each species, the following fluid equations (in e.s.u.):

$$mn \frac{\partial \mathbf{v}}{\partial t} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla \cdot \mathbb{P} \tag{1}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 . \tag{2}$$

Here the density n will be taken to be the same for ions and electrons, the mass will be written m for electrons and M for ions, and the charge q assumed to be $+e$ for ions and $-e$ for electrons. These equations are valid if the Larmor radii are much smaller than the scale of macroscopic gradients and if there are no ion-electron collisions. Finite resistivity will be treated in Sec. V . We further assume that the density is so low that $\nabla \times \mathbf{B} = 0$ and that the frequencies are so low that $\nabla \times \mathbf{E} = 0$; then $\mathbf{B}$ is constant and uniform and $\mathbf{E}$ can be written $-\nabla\phi$. If only frequencies much less than ω_{p_i} , the ion plasma frequency, are considered, we may omit Maxwell's equations altogether and use only Eqs.(1) and (2). We now assume that the viscosity may be neglected and that the temperatures are constant and uniform, so that

$$\nabla \cdot \mathbb{P} = KT \nabla n . \tag{3}$$

Eq.(1) is also valid in the limit of frequent collisions; the pressure and density are then related by an equation of state; Eq.(3) then contains a constant of order unity, but its nature is not changed.

For simplicity we shall consider a plane geometry, with the magnetic field $\underset{\sim}{B}$ lying in the z-direction, the zero-order density gradient in the x-direction, and the propagation vector $\underset{\sim}{k}$ in the y-z plane, its primary component being in the y-direction. If the time derivative and the electric field are assumed to vanish, Eq.(1) becomes, for ions,

$$e n_0 \underset{\sim}{v}^{(0)} \times \underset{\sim}{B} = K T_i n_0' , \tag{4}$$

where the prime denotes differentiation relative to x. Thus the equilibrium solution is

$$\begin{aligned} v_x^{(0)} &= 0 \\ v_y^{(0)} \equiv v_0 &= \frac{KT_i}{eB}\frac{n_0'}{n_0} = \frac{\lambda v_{th}^2}{\omega_c} , \end{aligned} \tag{5}$$

where

$$\lambda(x) \equiv n_0'/n_0, \quad v_{th}^2 \equiv KT_i/M, \quad \omega_c \equiv eB/M. \tag{6}$$

We now assume a perturbation of the form

$$\begin{aligned} & n = n_0 + n_1 , \quad n_1 = \nu n_0(x) \exp[i(k_\perp y + k_\parallel z - \omega t)] \\ & \underset{\sim}{E}^{(1)} = -\underset{\sim}{\nabla}\phi, \quad \phi = \bar{\phi} \exp[i(k_\perp y + k_\parallel z - \omega t)] \\ & \underset{\sim}{v} = v_0 \hat{y} + \underset{\sim}{v}^{(1)}, \quad \underset{\sim}{v}^{(1)} = \bar{\underset{\sim}{v}} \exp[i(k_\perp y + k_\parallel z - \omega t)], \end{aligned} \tag{7}$$

in which ν is a real constant by assumption, and $\bar{\phi}$ and $\bar{\underset{\sim}{v}}$ are therefore constant by virtue of $\underset{\sim}{\nabla} \times \underset{\sim}{E} = 0$ and the equations of motion. We have thus, for convenience, required the density perturbation to have the same x-dependence as n_o; but it will not be necessary to specify the form of $n_o(x)$. Eq.(1) for ions becomes

$$i\omega M n_o \underset{\sim}{v}^{(1)} + e(n_o+n_1)[-\underset{\sim}{\nabla}\phi + (\underset{\sim}{v}^{(0)}+\underset{\sim}{v}^{(1)}) \times \underset{\sim}{B}] - KT_i(\nabla n_o + \nabla n_1) = 0. \qquad (8)$$

Upon subtracting $(1+\nu)$ times Eq.(4) from this and separating into components, we obtain, as a consequence of (7),

$$\begin{aligned} i\omega M n_o \bar{v}_x + e n_o \bar{v}_y B &= 0 \\ i\omega M n_o \bar{v}_y &= e n_o (ik_\perp \bar{\phi} + v_x B) + ik_\perp KT_i \nu n_o \\ i\omega M n_o \bar{v}_z &= ik_\parallel n_o (e\bar{\phi} + KT_i \nu). \end{aligned} \qquad (9)$$

The solution of these gives

$$\begin{aligned} \bar{v}_x &= \frac{i}{\Omega^2 - 1} \frac{k_\perp}{eB} (e\bar{\phi} + \nu KT_i) \\ \bar{v}_y &= -i\Omega \bar{v}_x \\ \bar{v}_z &= \frac{k_\parallel}{eB\Omega} (e\bar{\phi} + \nu KT_i), \end{aligned} \qquad (10)$$

with

$$\Omega \equiv \omega/\omega_c . \qquad (11)$$

The equation of continuity, Eq.(2), becomes, when we linearize and assume that $v_z^{(o)}$ vanishes,

$$(\omega - k_\perp v_o)\nu - k_\perp \bar{v}_y - k_\parallel \bar{v}_z + i\bar{v}_x \omega_c v_o v_{th}^{-2} = 0. \tag{12}$$

The value of $\bar{\underset{\sim}{v}}$ can now be inserted from Eq.(10) to obtain a relation between $\bar{\phi}$ and ν. It will now be convenient to define the following dimensionless parameters :

$$\begin{aligned}
&\chi \equiv \frac{e\bar{\phi}}{KT_i}, \qquad \beta \equiv T_e/T_i, \qquad \mu = m/M \\
&\gamma \equiv k_\perp v_{th}/\omega_c = k_\perp r_L/\sqrt{2} \\
&\gamma_\parallel \equiv k_\parallel v_{th}/\omega_c = k_\parallel r_L/\sqrt{2} \\
&\delta \equiv \lambda v_{th}/\omega_c = \lambda r_L/\sqrt{2} \\
&\kappa \equiv k_\perp v_o/\omega_c = \gamma\delta, \qquad \theta = \gamma_\parallel/\gamma,
\end{aligned} \tag{13}$$

where $r_L = \sqrt{2}\, v_{th}/\omega_c$ is the ion Larmor radius, and the other symbols are defined in Eqs.(5) and (6). Note that λ and δ are not assumed to be constant. In terms of these parameters, the equation of continuity for the ions, as found from (10) and (12), is as follows :

$$\begin{aligned}
&[\Omega(\Omega^2-1)(\Omega-\kappa) - \gamma\Omega(\gamma\Omega+\delta) - \gamma_\parallel^2(\Omega^2-1)]\,\nu \\
&\qquad - [\gamma\Omega(\gamma\Omega+\delta) + \gamma_\parallel^2(\Omega^2-1)]\,\chi = 0.
\end{aligned} \tag{14}$$

To obtain the equivalent equation for electrons, we can simply replace Ω by $-\mu\Omega$, χ by $-\beta^{-1}\chi$, δ^2 by $\beta\mu\delta^2$, γ^2 by $\beta\mu\gamma^2$, $\gamma_{/\!/}^2$ by $\beta\mu\gamma^2\theta^2$, and κ by $\beta\mu\kappa$. This leads to the following continuity equation for electrons :

$$\left[\Omega^2\{\beta^{-1}(1-\mu^2\Omega^2)+\mu\gamma^2-\mu^2\Omega\kappa\}-\gamma^2\mu^{-1}\theta^2(1-\mu^2\Omega^2)\right]\beta\nu + \\ +\left[\gamma\Omega(\delta-\mu\gamma\Omega)+\gamma^2\mu^{-1}\theta^2(1-\mu^2\Omega^2)\right]\chi = 0. \tag{15}$$

We can simplify Eq.(15) by neglecting the term $\mu^2\Omega^2$ relative to 1; since Ω is less than or equal to 1 for the waves under consideration, this is an extremely good approximation. The term $\mu^2\Omega\kappa$ may also be neglected relative to $\mu\gamma^2$ if $\mu\Omega << \gamma/\delta$, which is easy to satisfy. In addition, we can simplify the ion equation (14) by neglecting the terms containing $\gamma_{/\!/}^2$. This means we neglect the motion of ions along $\underset{\sim}{B}$. Comparing the term $\gamma_{/\!/}^2$ to the term $\gamma^2\Omega^2$, we see that this approximation requires $\theta^2 << \Omega^2$, which is valid for the range of interest. Eqs. (14) and (15) then become

$$\Omega\left[(\Omega^2-1)(\Omega-\kappa)-(\gamma^2\Omega+\kappa)\right]\nu-\Omega(\gamma^2\Omega+\kappa)\chi=0 \tag{14'}$$

$$\left[\Omega^2(\beta^{-1}+\mu\gamma^2)-\gamma^2\mu^{-1}\theta^2\right]\beta\nu+\left[\Omega(\kappa-\mu\gamma^2\Omega)+\gamma^2\mu^{-1}\theta^2\right]\chi=0 \tag{15'}$$

The requirement that the determinant of these two equations vanishes then supplies the dispersion relation.

We note in passing that if the inertia term in Eq.(1) had been neglected, as is done in analyses of the screw-instability type, Eqs.(14') and (15) would have been identical; and no condition on Ω would have been found, the equation of continuity being satisfied identically for each species separately. Thus the inertia of at least the ions must be taken into account, and the effects we shall consider depend on the fact that the ion Larmor radius, through small, is finite.

III - PERPENDICULAR PROPAGATION.

We first consider the limiting case in which $\theta = 0$, and the problem is entirely two-dimensional. The electron equation (15') then becomes simply

$$\Omega(1+\beta\mu\gamma^2)\nu + (\kappa-\mu\gamma^2\Omega)\chi = 0. \tag{16}$$

Together with the ion equation (14'), this leads to the dispersion equation

$$\Omega^2(\kappa-\mu\gamma^2\Omega)[\Omega^2-\kappa\Omega-(1+\gamma^2)] + \Omega^2(\kappa+\gamma^2\Omega)(1+\beta\mu\gamma^2) = 0. \tag{17}$$

1. <u>Limit $\lambda \to 0$.</u> In the case of a uniform plasma, κ vanishes, and for $\Omega \neq 0$ Eq. (17) reduces to

$$\mu[\Omega^2-(1+\gamma^2)] = 1+\beta\mu\gamma^2, \tag{18}$$

or, for $\mu \ll 1$,

$$\Omega^2 = \mu^{-1} + (1+\beta)\gamma^2 \tag{19}$$

$$\omega^2 = \omega_{ce}\omega_c + k_\perp^2 v_s^2 , \tag{20}$$

where

$$v_s = (KT_i + KT_e)^{\frac{1}{2}} / M^{\frac{1}{2}} \tag{21}$$

is the acoustic velocity. We thus recover the lower hybrid frequency for perpendicular propagation in a uniform plasma.

2. Limit $B \to 0$. In the limit of vanishing magnetic field, we must also assume $\lambda = 0$ in order to have an equilibrium. We can then simply set $\omega_{ce}\omega_c$ equal to zero in Eq.(20) and recover the expression for ordinary acoustic waves in the limit of low frequencies.

3. Limit $\mu \to 0$. In the limit of vanishingly small electron Larmor radius, we may set μ equal to zero in Eq.(17). The equation can then be factored exactly :

$$\Omega^2 (\Omega^2 - \kappa)(\Omega + \gamma\delta^{-1}) = 0. \tag{22}$$

This gives three roots :

$$\begin{aligned} &\text{a)} \quad \omega = 0 \\ &\text{b)} \quad \omega = k_\perp v_0 \\ &\text{c)} \quad \omega = (-k_\perp/\lambda)\,\omega_c . \end{aligned} \tag{23}$$

The first root, a double one at $\omega = 0$, corresponds to the most obvions physical situation : an equal number of ions and electrons is added to the equilibrium configuration, and these simply gyrate in place without causing any charge separation. It is seen from Eq.(16) that if Ω vanishes, so does the electric field. Referring back to the ion equations of motion (10), we see that v_y vanishes for this root, and

$$\bar{v}_x = -ik_\perp \frac{KT_i}{eB} \nu . \tag{24}$$

This is just the ∇p drift of the perturbation; according to the physical picture given by Spitzer[6], there is a net velocity in the x-direction, even though all guiding centers remain fixed, because of the difference in density of guiding centers in the y-direction.

The second root, $\omega = k_\perp v_0$, is the one discussed by D'Angelo[1,2], and is a wave which travels with the zero-order drift velocity. Referring to Eq.(16), we see that for $\mu = 0$ this root gives a finite electric field :

$$\chi = -\Omega\nu/\kappa = -\nu . \tag{25}$$

Inserting this into the equations of motion (10), we see that in this mode $v_x = v_y = 0$, and the ion macroscopic velocity vanishes. This is because the pressure gradient drift of the ion perturbation, which was found for the static case $\Omega = 0$, is in this case exactly canceled by a drift of the ion guiding centers in the x-direction, caused by the oscillating electric field in the y-direction. This is illustrated in Fig. 1. This

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[6] L. Spitzer - Physics of Fully Ionized Gases (Interscience Publishers, New York, 1962), 2nd ed., p.32.

E-field also causes an $\underset{\sim}{E} \times \underset{\sim}{B}$ drift of the electrons; and if ion inertia had been neglected, this drift would have been the same for the ions, and there would have been no charge separation to maintain the E-field. However, since the ions move slightly differently from the electrons under the action of E~~y~~, the electrons must drift back and forth in the x-direction to maintain charge neutrality at all times. The density gradient in the x-direction is, of course, what makes this mechanism of neutralization possible. This wave, therefore, is by no means trivial; the ion and electron guiding centers may drift over large distances because of the low frequency involved.

From this physical picture it is easy to see intuitively that the velocity of the wave should depend on the speed with which the electrons can neutralize the space charge; that is, it should be proportional to the local density gradient λ and the drift speed E/B. Since E/B must be proportional to KT_i/B in order for the ion drifts to cancel, we have $\omega/k_\perp \sim KT_i\lambda/B = v_0$. Why the constant of proportionality should be exactly 1 seems to be fortuitous. Indeed, if we had chosen a different x-dependence for n_1, the wave velocity would be slightly different from v_0. Although we have not considered propagation in the x-y plane, it is clear that the wave velocity will be fastest in the y-direction, since the $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts are then parallel to the density gradient.

We now consider the root (c) of Eq. (23). This is a modified ion cyclotron wave in which electrons maintain quasineutrality not by moving along $\underset{\sim}{B}$ as in the usual case but by drifting along a density gradient. Whereas in solutions (a) and (b) the continuity equation for electrons and ions, respectively, was satisfied identically because the macroscopic velocity was zero, in solution (c) both velocities are finite. The ratio of wave velocities for (b) and (c) is $-\delta^2 = \frac{-1}{2}\lambda^2 r_L^2$, which, by assumption, is very small. We note that the frequency of wave (c) can be larger or smaller than ω_c depending on the relative size of the wavelength and the density gradient, and that the wave always travels in the opposite direction from v_0.

4. Case $\mu = 1$. For completeness we finally consider the solution of Eqs.(14) and (15) for the case of a positronium or solid-state plasma where $\mu = \beta = 1$. For strictly perpendicular propagation, these equations yield without approximation the simple dispersion relation

$$\omega^2 = \omega_c^2 + (k_\perp^2 - \lambda^2) v_{th}^2 , \tag{26}$$

which shows the continuous transition from what corresponds to the lower hybrid frequency of Eq.(20) to what corresponds to the root (c) of Eq.(23) as $|\lambda|$ is increased from 0 to $|k|$. The drift wave cannot arise because the Larmor radii are equal.

IV - PROPAGATION AT "LARGE" ANGLES.

We now proceed to the opposite limit of Eq.(15'), in which $\mu^{-1}\theta^2 \to \infty$. Eq. (15') then reduces to the (linearized) Boltzmann distribution for the electrons :

$$\chi = \beta\nu, \tag{27}$$

Since the electrons have been assumed to move freely along $\underset{\sim}{B}$. Insertion of this into the ion equation (14') yields the cubic dispersion equation

$$(\Omega-\kappa)(\Omega^2-1) - (1+\beta)(\gamma^2\Omega+\kappa) = 0. \tag{28}$$

In the limit of a uniform plasma, $\kappa = 0$, and we recover the dispersion relation for ion cyclotron waves, propagating perpendicular to $\underset{\sim}{B}$, found by Motley and d'Angelo[3]:

$$\Omega^2 = 1 + (1+\beta)\gamma^2, \qquad \omega^2 = \omega_c^2 + k_\perp^2 v_s^2 . \tag{29}$$

The small correction $k_\perp^2 v_s^2$ is somewhat simpler in form than that found by Drummond and Rosenbluth[4].

When the density gradient is finite, the dispersion equation (28) may be solved by considering it a quadratic in the normalized wavelength γ ; since $\kappa = \delta\gamma$, we have

$$(1+\beta)\Omega\gamma^2 + (\Omega^2+\beta)\delta\gamma + \Omega(1-\Omega^2) = 0. \tag{30}$$

This is shown plotted on Fig. 2 for $\beta = 1$ and $\delta = 0, 0.1$, and 0.5 . It is clear from (30) that the graph is symmetric about the origin; therefore, there are always three real roots Ω for each value of γ . For $\delta = 0$, the parabola represents the relation (29). When δ is finite, this branch is lowered and moved to the left; thus for a given wavelength, the frequency is raised or lowered by the density gradient depending on the sign of $\gamma\delta$. (Note, however, that this effect is not directly applicable to experiments[3] on cyclotron waves in which the density gradient is parallel to $\underset{\sim}{k}$). In addition, a low-frequency density-gradient wave appears for finite δ. For this wave, we can neglect the term Ω^2 in Eq. (28); the frequency is then approximately

$$\Omega = -\beta\kappa\left[1 + (1+\beta)\gamma^2\right]^{-1} \approx -\beta\kappa. \tag{31}$$

From Eq.(5), it is clear that this wave travels at approximately the <u>electron</u> drift velocity. This is similar to the drift wave of the previous section, but the direction of propagation is reversed and the ion temperature is replaced by the electron temperature. The latter is because the electric field is now determined by the electron temperature, as Eq.(27) clearly shows. The fact that the density gradient enters with the opposite sign is because now that the electrons do not move across $\underset{\sim}{B}$ at all, only the ions are affected by the mechanism of charge neutralization via $\underset{\sim}{E} \times \underset{\sim}{B}$ drifts along a density gradient.

V - ARBITRARY ANGLES : FINITE ELECTRON INERTIA.

To see the transition between the two sets of waves found in Secs.III and IV, we must take into account the mechanism which limits electron motion along $\underset{\sim}{B}$. In this section we assume that collisions are unimportant, and the electron streaming velocity is limited only by inertia. Since this inertia is small, the effect will be appreciable only at very small values of θ, corresponding to very long wavelengths along $\underset{\sim}{B}$. Thus we must solve the determinantal equation of (14') and (15'), assuming that μ is small but $\mu^{-1}\theta^2$ is finite. The resulting dispersion relation is quartic in Ω and cubic γ, but it is linear in $\mu^{-1}\theta^2$, and we can regard it as a relation between $\mu^{-1}\theta^2$ and Ω, for a given real value of γ. After a little algebra, we obtain, correct to first order in μ,

$$\mu^{-1}\theta^2 = \frac{\Omega^2(\Omega-K)(\delta\Omega+\gamma)\gamma^{-1}}{(1-\Omega^2)(\Omega-K)+(1+\beta)(\gamma^2\Omega+K)} + \mu\Omega^2. \tag{32}$$

Aside from the small term $\mu\Omega^2$, this expression reduces to the relation (22) of Sec.III if θ^2 is allowed to go to zero faster than μ, and to the relation (28) of Sec.IV if μ is allowed to go to zero faster than θ^2. In the limit $\delta = K = 0$, this reduces to

$$\Omega^2 = \frac{1+(1+\beta)\gamma^2}{1+\mu\theta^{-2}}, \tag{33}$$

and Ω is always real, as it has to be, if the plasma is uniform. If is finite, it will be convenient to define a normalized wave velocity

$$u \equiv \Omega/K, \tag{34}$$

and to write Eq.(32) (without the $\mu\Omega^2$ term) in terms of u :

$$\mu^{-1}\theta^2 = \frac{u^2(u-1)(1+\delta^2 u)}{K^{-2}(u+1)+2\delta^{-2}u-u^2(u-1)}. \qquad (\beta=1) \tag{35}$$

This relation between the wave velocity u and the angle of propagation θ is shown in Fig.3a for a case $|\delta| < \gamma$ and in Fig. 3b for a case $|\delta| > \gamma$. The sign of δ does not enter. The intersections with the u-axis are the three waves of Sec.III; the asymptotes are the three waves of Sec.IV. It is clear from Fig.3 that the transition occurs for very small angles θ, and that the ion drift wave at $u = 1$ has already greatly changed its velocity when θ is only as large as $0.1\ \mu^{\frac{1}{2}}$.

For small values of $\mu^{-1}\theta^2$, two real roots of u are lost, indicating the possibility of an instability. This is not a consequence of our approximations, since the term $\mu\Omega^2$ in (32) is small even in the unstable region, and the approximations made to obtain (14') and (15') are valid for small $\mu^{-1}\theta^2$. In the case of Fig.3a, apparently the electron drift wave speeds up and the "backward" cyclotron wave slows down as θ is decreased, and these two waves interact at the same velocity to give an instability. We have not shown the imaginary part of u because in practice this phenomenon will be masked by the effects of resistivity.

VI - ARBITRARY ANGLE : FINITE RESISTIVITY.

We now wish to consider the more realistic case where the electron velocity is limited by collisions rather than by inertia. For this it is more convenient to use the single-fluid equations[7] for a fluid

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[7] L.Spitzer - Physics of Fully Ionized Gases (Interscience Publishers, New-York, 1962) 2nd ed. p.27-28.

with a macroscopic velocity $\underset{\sim}{v}$, a macroscopic current $\underset{\sim}{j}$, and a resistivity η; thus the set of four equations (1) and (2) in $\underset{\sim}{v}_i$ and $\underset{\sim}{v}_e$ is replaced by the following set in $\underset{\sim}{v}$, $\underset{\sim}{j}$:

$$Mn \frac{\partial \underset{\sim}{v}}{\partial t} = \underset{\sim}{j} \times \underset{\sim}{B} - (1+\beta) KT_i \underset{\sim}{\nabla} n \tag{36}$$

$$n \underset{\sim}{v} \times \underset{\sim}{B} = n \underset{\sim}{\nabla} \varphi + n\eta \underset{\sim}{j} + \frac{1}{e} \underset{\sim}{j} \times \underset{\sim}{B} - \frac{\beta K T_i}{e} \underset{\sim}{\nabla} n \tag{37}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{j} = 0 \tag{38}$$

$$\frac{\partial n}{\partial t} + \underset{\sim}{\nabla} \cdot (n \underset{\sim}{v}) = Q = q - \alpha n^2 . \tag{39}$$

In Eq.(37) we have neglected a term in $\partial \underset{\sim}{j}/\partial t$ which represents the electron inertia; this requires that the resistivity be high enough that the electron collision frequency is much higher than the wave frequency, a condition easily satisfied in practice. We have also neglected terms of order m/M. Because the plasma is no longer perfectly confined, we have had to add a source term Q to (39) in order to achieve an equilibrium.

In zero-order, we assume that $\partial/\partial t = \partial/\partial\theta = \partial/\partial z = \underset{\sim}{\nabla}\phi = 0$; the equations of motion (36) and (37) then give, in cylindrical coordinates,

$$\begin{aligned} j_r^{(0)} &= 0 \\ j_\theta^{(0)} &= \frac{(1+\beta)KT_i}{B} n_0' \\ v_\theta^{(0)} &= \frac{KT_i}{eB} \frac{n_0'}{n_0} \\ v_r^{(0)} &= -\frac{\eta}{B} j_\theta^{(0)} = -A n_0' , \end{aligned} \tag{40}$$

where the prime denotes $\partial/\partial r$, and A is defined by

$$A = \frac{(1+\beta)\eta K T_i}{B^2} . \tag{41}$$

Eq.(38) is identically satisfied, but to satisfy (39), n_o cannot be an arbitrary function of r . If Q were proportional to n_o , Eq.(39) would be non-linear in n_o . However, in thermally-ionized plasmas, Q can have the form of a constant q minus a recombination term αn_o^2. In such a case (39) is linear in n_o^2 , and the solution for a cylinder of radius r_o is

$$n_o^2 = \frac{q}{\alpha}\left[1 - \frac{I_o[r(2\alpha/A)^{\frac{1}{2}}]}{I_o[r_o(2\alpha/A)^{\frac{1}{2}}]}\right], \tag{42}$$

where I_o is the usual Bessel function of imaginary argument. This profile was first found by Rynn and D'Angelo[8]. We shall assume that A is small, so that q and α are small; then we may neglect Q in first-order relative to $\partial n/\partial t$. This requires that η be sufficiently small that we can neglect diffusion during the period of an oscillation, but sufficiently large that the term $\partial j/\partial t$ can be neglected. These requirements are easily satisfied in practice.

We may then proceed to calculate the dispersion relation in the same manner as in Sec.V. The algebra is somewhat more complicated and will be omitted. Upon taking the same form for the perturbation (Eq.7) and again neglecting the ion motion parallel to $\underset{\sim}{B}$, we obtain

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[8] N. Rynn and N. D'Angelo - Rev.Sci.Instr. 31, 1326 (1960).

$$-i\epsilon^{-1}\theta^2 = \frac{\Omega(\Omega-\kappa)(\delta\Omega+\gamma)\gamma^{-1}}{(1-\Omega^2)(\Omega-\kappa)+(1+\beta)(\gamma^2\Omega+\kappa)}, \tag{43}$$

in which

$$\epsilon \equiv \frac{n_o e\eta}{B} \tag{44}$$

has been assumed to be small.

The form of Eq.(43) is not surprising, for if the z-component of (37) is compared with the z-component of (1) for electrons, one finds that $-i\omega mnv_{ez}$ has been replaced by $-ne\eta j_z$. Since ion motion in the z-direction has been neglected, j_z is merely $-en_o v_{ez}$. Dividing through by eB, we find that $-i\Omega\mu$ is to be replaced by ϵ, or μ^{-1} by $-i\Omega\epsilon^{-1}$. This explains the factor of $-i\Omega$ which appears between Eq.(32) and Eq.(43). This factor not only changes the degree of the equation but also makes the coefficients complex so that Ω is always complex and must be found numerically.

In the limit $\delta = \kappa = 0$, Eq.(43) reduces to

$$i\epsilon^{-1}\theta^2(\Omega^2 - C) = \Omega, \tag{45}$$

with $$C = 1 + (1+\beta)\gamma^2.$$

Upon separating Ω into real and imaginary parts and taking the real part of Eq.(45), we find that $\mathrm{Im}(\Omega) = -\frac{1}{2}\epsilon\theta^{-2} < 0$. Thus in a uniform plasma the waves are damped, as one would expect. For finite δ, when $\epsilon^{-1}\theta^2 \to 0$, the numerator of Eq.(43) must vanish, and we recover Eq.(22); when $\epsilon^{-1}\theta^2 \to \infty$, the denominator must vanish, and we recover

Eq.(28). In terms of the normalized wave velocity u, Eq.(43) can be written

$$i\epsilon^{-1}\theta^2 = \frac{\kappa u(u-1)(1+\delta^2 u)}{\kappa^2 u^2(u-1) - Cu - \beta} . \tag{46}$$

This relation has been computed numerically for $\beta = 1$ and γ real. In Fig.4 is shown a case with the same parameters as in Fig.3; in Fig.5, a case corresponding to a thermally-ionized potassium plasma; and in Fig. 6, a case corresponding to a stellarator plasma. Only positive values of $\epsilon^{-1}\theta^2$ are shown because no pathological behavior occurs near the u-axis, as in Fig.3. From (46) it is evident that when $\epsilon^{-1}\theta^2$ changes sign, u goes into its complex conjugate.

We note that although the imaginary part of u (or of Ω, since κ is real) goes to zero for large and small $\epsilon^{-1}\theta^2$, in accord with the results of Sec.III and IV, it is finite and appreciable for a large range of $\epsilon^{-1}\theta^2$. Since the sign of κ affects Eq.(46), we have chosen it positive so that positive Im(u) corresponds to positive Im(Ω). We then see that the density-gradient wave traveling with the electrons is unstable for $\epsilon^{-1}\theta^2 \sim 1$, and that the "forward" wave (traveling in the ion drift direction) is always damped. The "backward" ion cyclotron wave is excited if $\delta > \gamma$ and damped if $\delta < \gamma$. The physical reason for the rather surprising appearance of positive growth rates lies in the correlation between the ion velocity v_x and the density perturbation ν. From Eq.(10) we see that if $\overline{\phi}$ and ν are in phase, as in Eq.(25) or (27), v_x is 90° out of phase with ν, and there can be no growth. The flow of electrons along $\underset{\sim}{B}$ shifts the phase of $\overline{\phi}$ and ν, and for certain waves there can be a correlation between v_x and ν such that v_x brings more density into the perturbation from the zero-order distribution when ν is already positive. The perturbation can then grow at the expense of the zero-order gradient.

VII - DISCUSSION.

In this paper we have overlooked the effects of radial boundary conditions, viscosity, and zero-order drifts along $\underset{\sim}{B}$. Since it has been shown that only the local density gradient matters for small amplitude waves, the radial boundaries will not greatly affect waves propagating in the azimuthal direction. Viscosity effects would not be important unless large shear velocities appear. A small drift of the electrons relative to the ions in zero-order could greatly affect the growth rate but would have a minor effect on the frequency. With these reservations, we now discuss the relevance of the results to experiments.

In the potassium plasma experiment of D'Angelo and Motley[1], ϵ is of the order of 3×10^{-5} and $\Omega\mu$ of the order of 3×10^{-7} , so that resistivity is about 10^{2} times more important than electron inertia, and the results of Sec.VI are relevant. In Fig.5 are shown the curves for a value of γ corresponding to such a plasma. It is apparent that for the "forward" wave $\epsilon^{-1}\,\theta^{2}$ must be below 10^{-1} for the wave velocity to be close to the ion drift velocity. This implies a longitudinal wavelength in excess of 5×10^{3} cm. If the wave is excited by an electron drift, the drift velocity must be of the order of 5×10^{7} cm/sec in order to be synchronous with the wave at 10 kc. To drive such a current, a potential drop ot the order of 10 volts along the plasma column would be necessary, and this seems excessive in view of the fact that no voltage is directly applied. Thus it is difficult to understand how the ion drift wave can be excited. On the other hand, the wave traveling with the electron drift maintains its frequency for angles θ which are compatible with the length of the column, and this wave can apparently be excited by the pressure gradient alone. It remains to explain why this wave appears only when an ion sheath is present at both ends of the plasma column. We suggest that perhaps it has to do with the insulating properties of the sheath. When the electron emission is limited by a potential hill, a rise in plasma potential allows more electrons to be emitted; thus electric fields are shorted out. However, when electron emission is temperature limited, it is independent of potential, and wavelengths longer than the machine are possible in the plasma column. Finally, we note that the "backward" cyclotron wave is not observed experimentally. This may be because the density gradient was not large enough to excite it, or because this macroscopic theory cannot accurately predict excitation close to ω_c.

It is interesting to see whether the "backward" density-gradient wave can be excited in a stellarator. Here also the resistivity dominates; the curves of Fig. 6 show a case with $\gamma = .01$, corresponding to conditions in a stellarator. The imaginary part of u is too small to be seen on the graph, but it is of the order of 0.1 to 0.5 times the real part of u at very low values of $\epsilon^{-1}\theta^2$. If $B \sim 30$ kgauss, $KT_e \sim 10$ eV, and both the wavelength and the scale length of the density gradient are of the order of 1 cm, the frequency ω is about 30 kc, and the growth time about 50-300μsec. However, extremely long wavelengths along $\underset{\sim}{B}$ are required; this is compatible with the requirements of the rotational transform, which must not be periodic. Thus it is not impossible for these waves to arise in a stellarator and to cause anomalous diffusion either by causing large amplitude drifts in the radial direction so that the outer layers are continually "scraped off", or by causing a state of turbulence in which particles can be lost by random walks.

VIII - ACKNOWLEDGMENTS.

This work was supported partly by the U.S. Atomic Energy Commision and partly by EURATOM. The hospitality of the Centre d'Etudes Nucléaires at Fontenay-aux-Roses, particularly in regard to help with the numerical computations, is gratefully acknowledged.

FIGURE CAPTIONS.

1. Pictorial representation of the density-gradient wave traveling with the ion drift velocity. The shaded region represents a density perturbation propagating in the y-direction, and the thickness of the ion Larmor orbits represents the relative local density. In this wave the $E \times B$ drifts in the x-direction exactly cancel the macroscopic velocity due to the perturbation.

2. Dispersion curves for waves traveling perpendicular to a density gradient, characterized by δ, when electron motion along the magnetic field is free. The graph is symmetric about the origin. Note the change of scale necessitated by the low frequency of the density-gradient wave.

3. The variation of the wave velocity u with angle of propagation θ, for a given wavelength γ and two values of the density gradient δ, in the case of zero resistivity. The value $\gamma = 0.18$ corresponds, for example, to the $m = 1$ mode of a thermally ionized Cs plasma 0.5 cm in radius at 5 kgauss or the $m = 3$ mode of a K plasma 1 cm in radius at 4 kgauss. Note that the scales are linear in the box near the origin and logarithmic or semi-logarithmic outside.

4. The variation of the wave velocity Re(u) and the growth rate Im(u) as a function of angle of propagation θ, in the case of finite resistivity, for the same values of γ and δ as in Fig.3. Note that the scales are linear in the box near the origin and logarithmic or semi-logarithmic outside.

5. The variation of Re(u) and Im(u) with θ, in the case of finite resistivity, for two values of δ and a value of γ corresponding to the $m = 1$ mode of a thermally-ionized K plasma 1.5 cm in radius at 3.5 kgauss. Note that the scales are linear in the box near the origin and logarithmic or semi-logarithmic outside.

6. The variation of Re(u) and Im(u) with θ, in the case of finite resistivity, for two values of δ and a value of γ corresponding to the $m = 2$ mode of a 2 eV helium plasma 2 cm in radius at 30 kgauss. Note that the scales are linear in the box near the origin and logarithmic or semi-logarithmic outside.

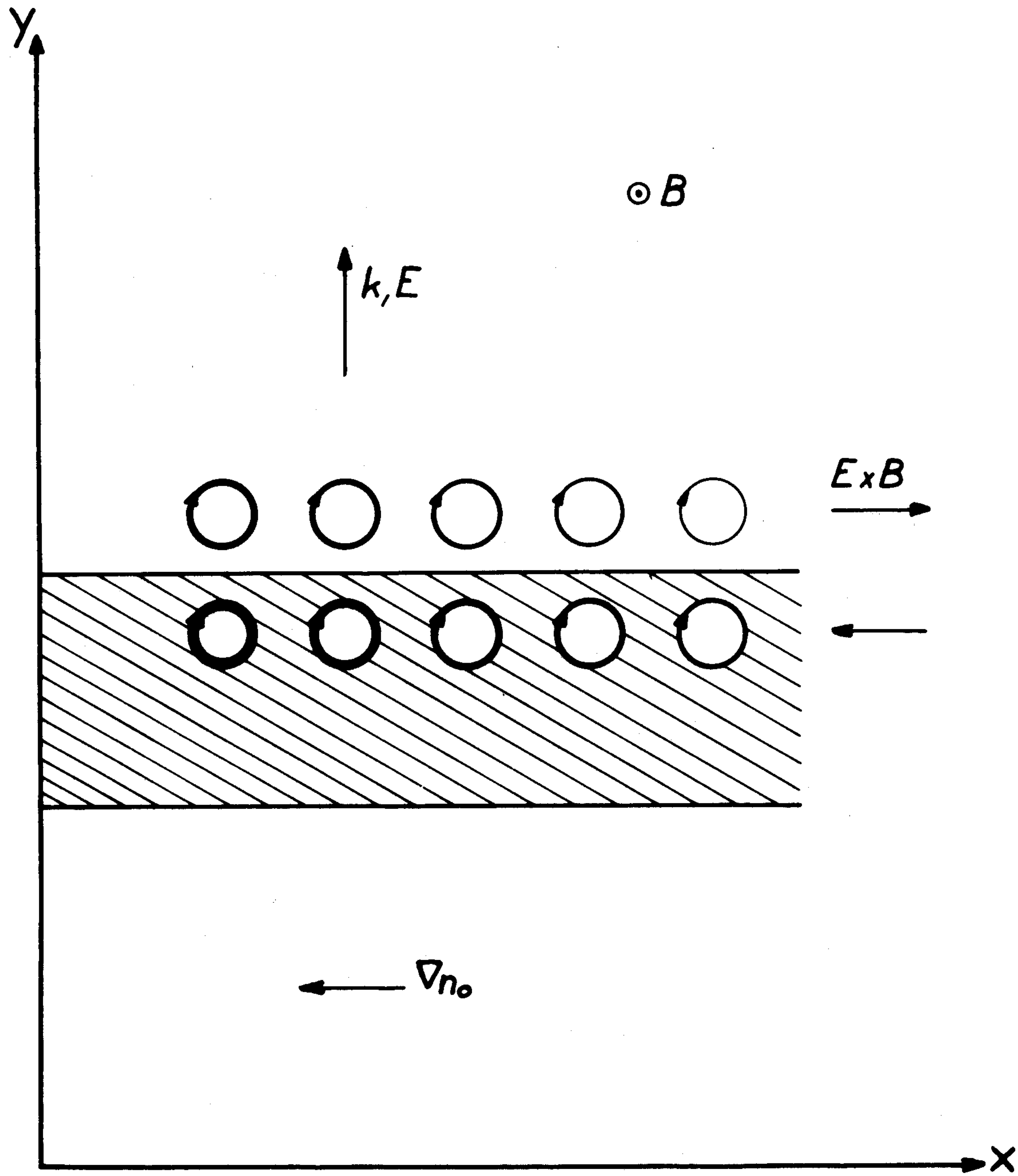
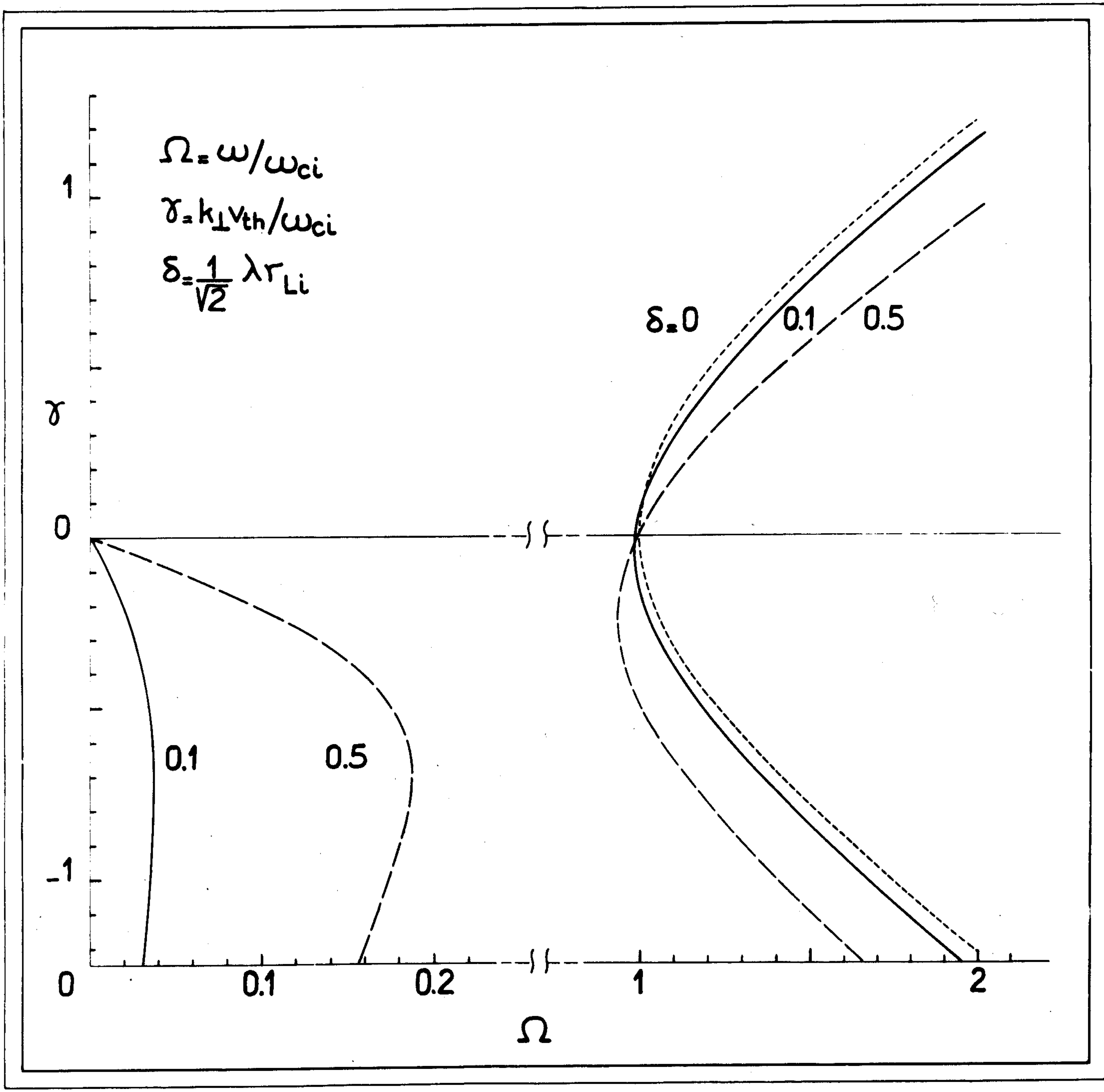
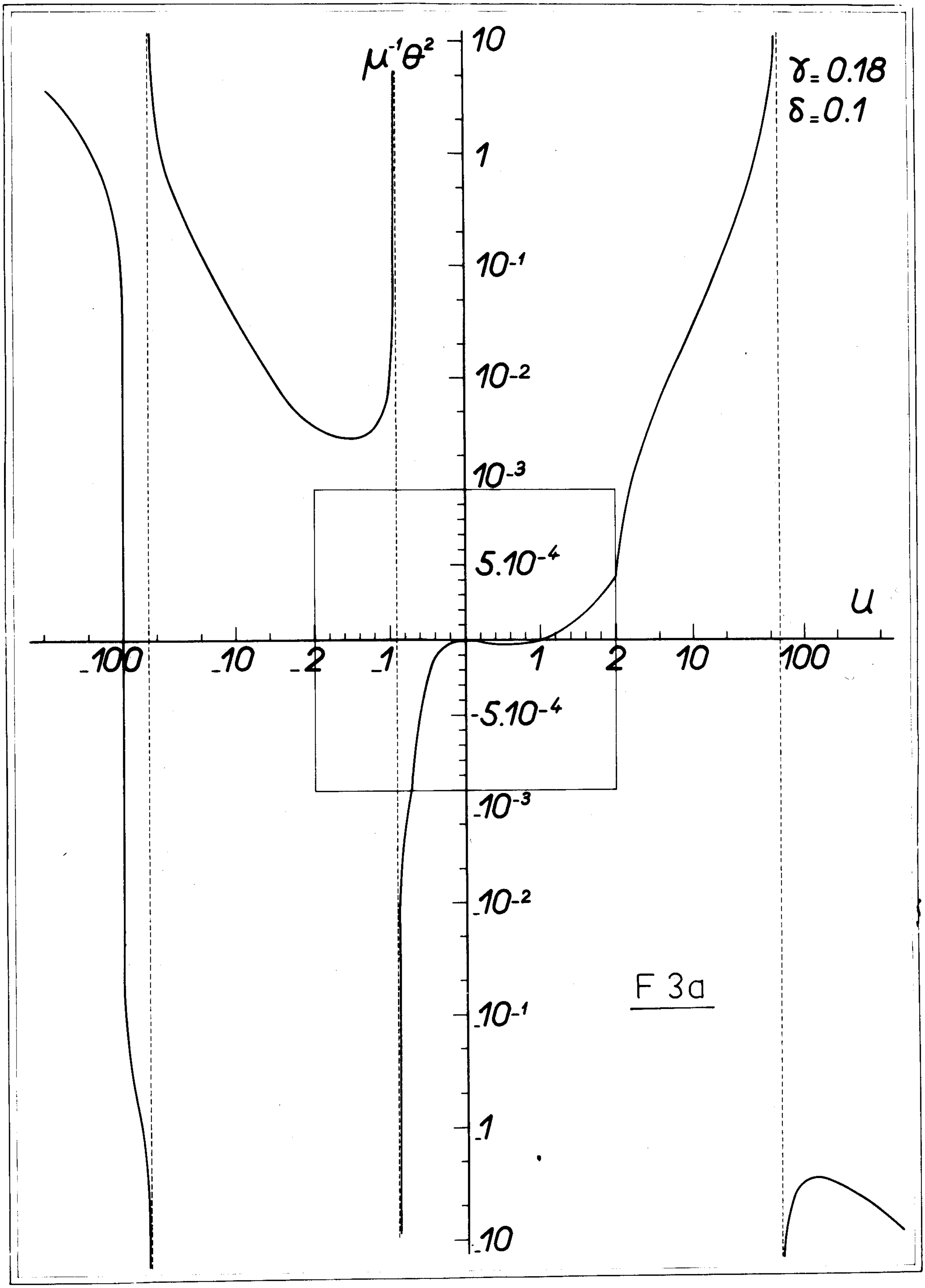


Figure: 1

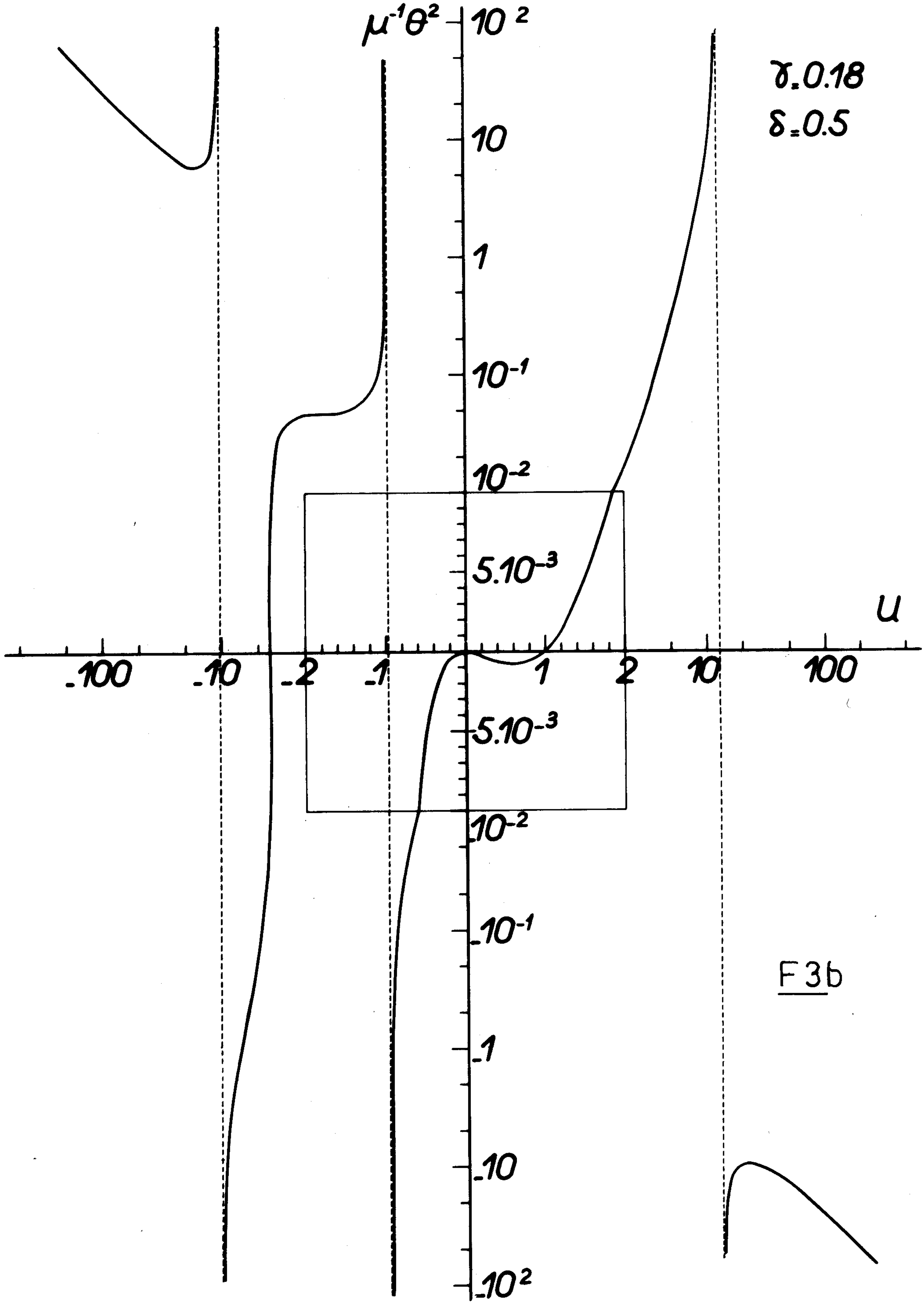

$\Omega = \omega/\omega_{ci}$
$\gamma = k_\perp v_{th}/\omega_{ci}$
$\delta = \frac{1}{\sqrt{2}} \lambda r_{Li}$
$\delta = 0$
0.1
0.5
0.1
0.5
1
0
-1
γ
0
0.1
0.2
1
2
Ω

F 2

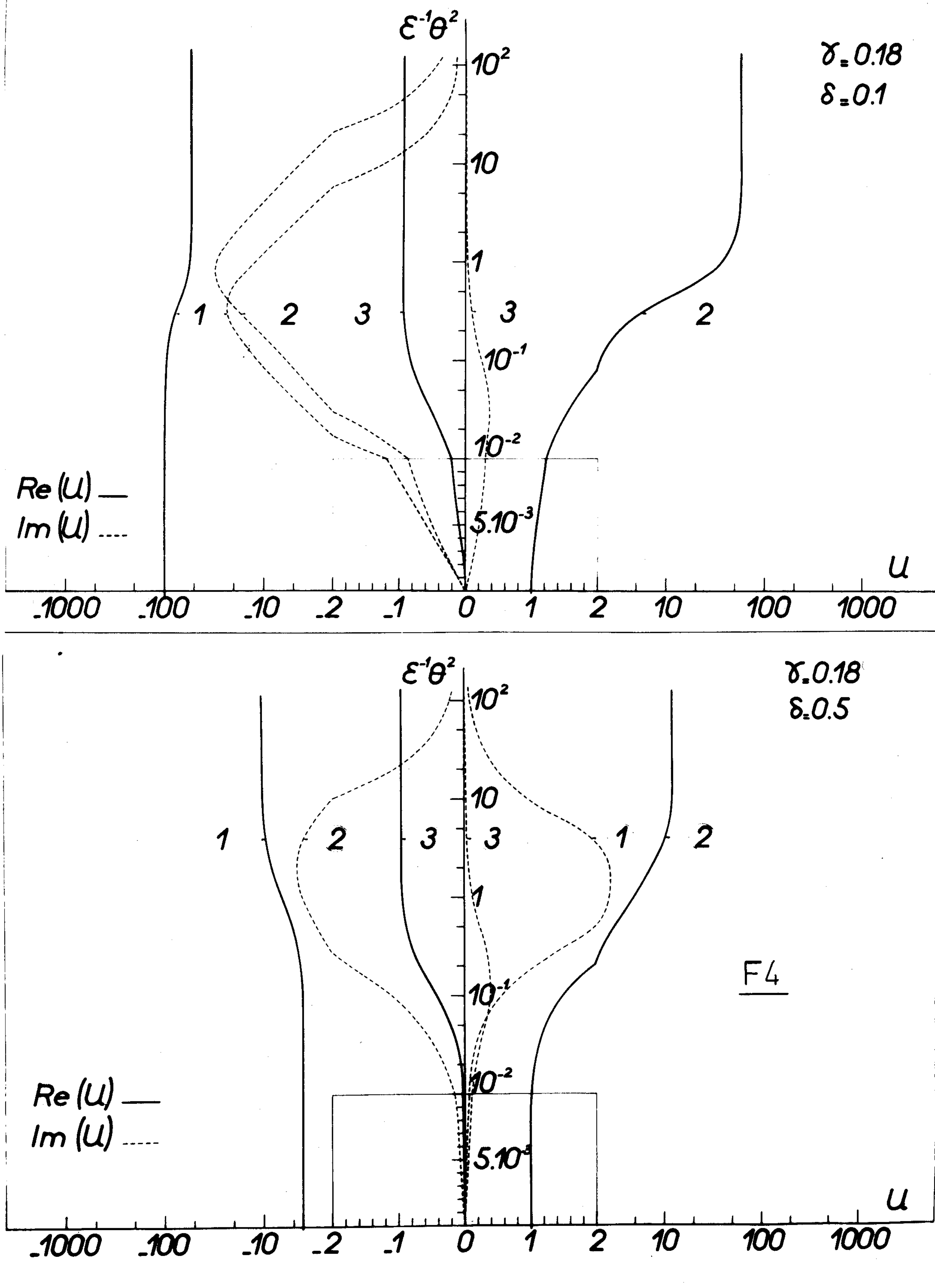


$\mu^{-1}\theta^2$
10
1
10-1
10-2
10-3
5.10-4
-5.10-4
-10-3
-10-2
-10-1
-1
-10
u
-100
-10
-2
-1
1
2
10
100
γ=0.18
δ=0.1

F 3a


$\mu^{-1}\theta^2$
10^2
10
1
10^{-1}
10^{-2}
5.10^{-3}
-5.10^{-3}
10^{-2}
-10^{-1}
-1
-10
-10^2
u
-100
-10
-2
-1
1
2
10
100
$\gamma = 0.18$
$\delta = 0.5$

F 3b



ε⁻¹θ²
γ = 0.18
δ = 0.1
Re(U) ——
Im(U) ----
1
2
3
3
2
10²
10
1
10⁻¹
10⁻²
5.10⁻³
U
-1000
-100
-10
-2
-1
0
1
2
10
100
1000
ε⁻¹θ²
γ = 0.18
δ = 0.5
Re(U) ——
Im(U) ----
1
2
3
3
1
2

F4

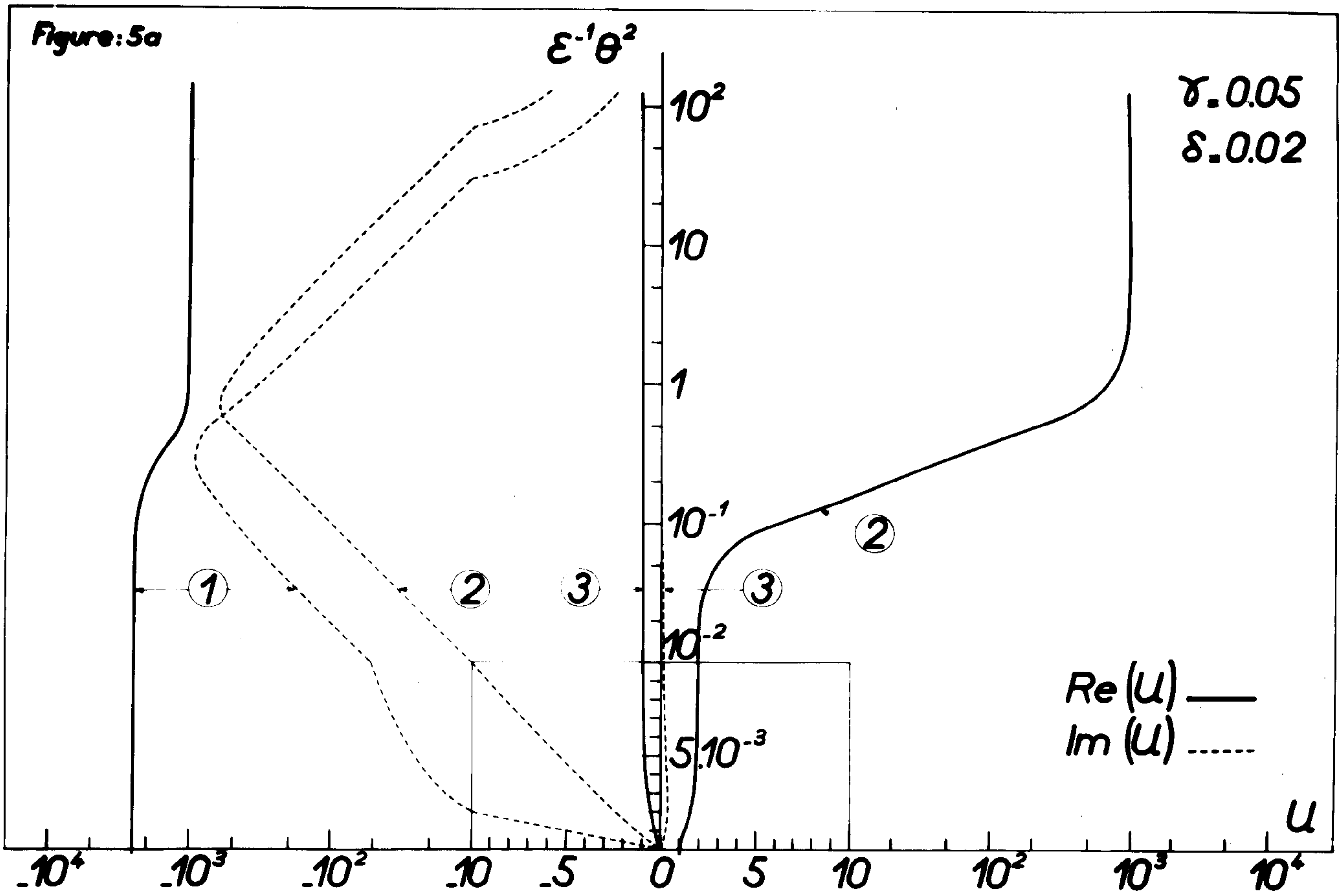


Figure: 5a
$\varepsilon^{-1}\theta^2$
$\gamma = 0.05$
$\delta = 0.02$
10^2
10
1
10^{-1}
10^{-2}
5.10^{-3}
1
2
3
3
2
Re (U)
Im (U)
U
-10^4
-10^3
-10^2
-10
-5
0
5
10
10^2
10^3
10^4

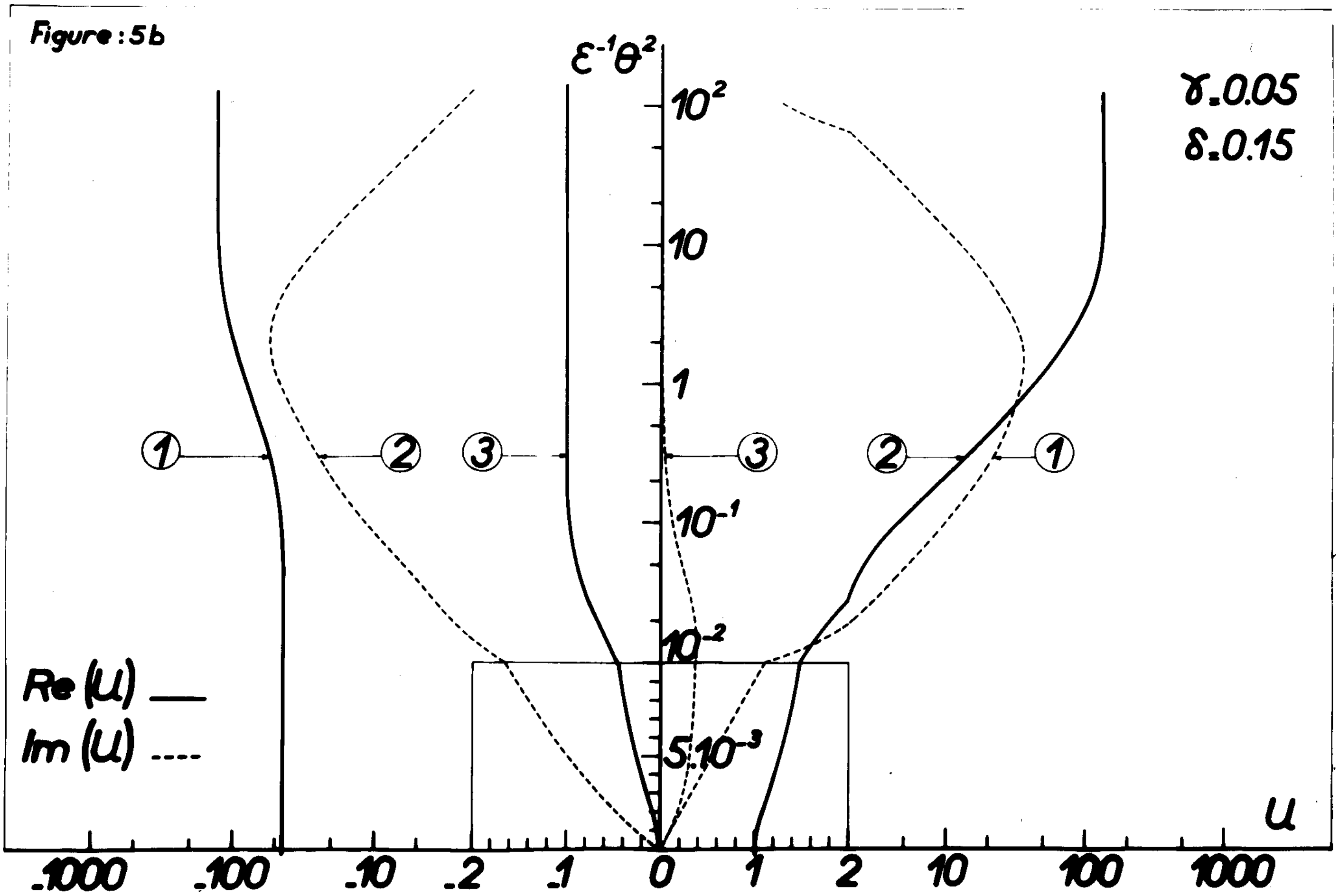


Figure: 5b
$\varepsilon^{-1}\theta^2$
$\gamma = 0.05$
$\delta = 0.15$
10^2
10
1
10^{-1}
10^{-2}
5.10^{-3}
1
2
3
3
2
1
Re (U)
Im (U)
U
-1000
-100
-10
-2
-1
0
1
2
10
100
1000

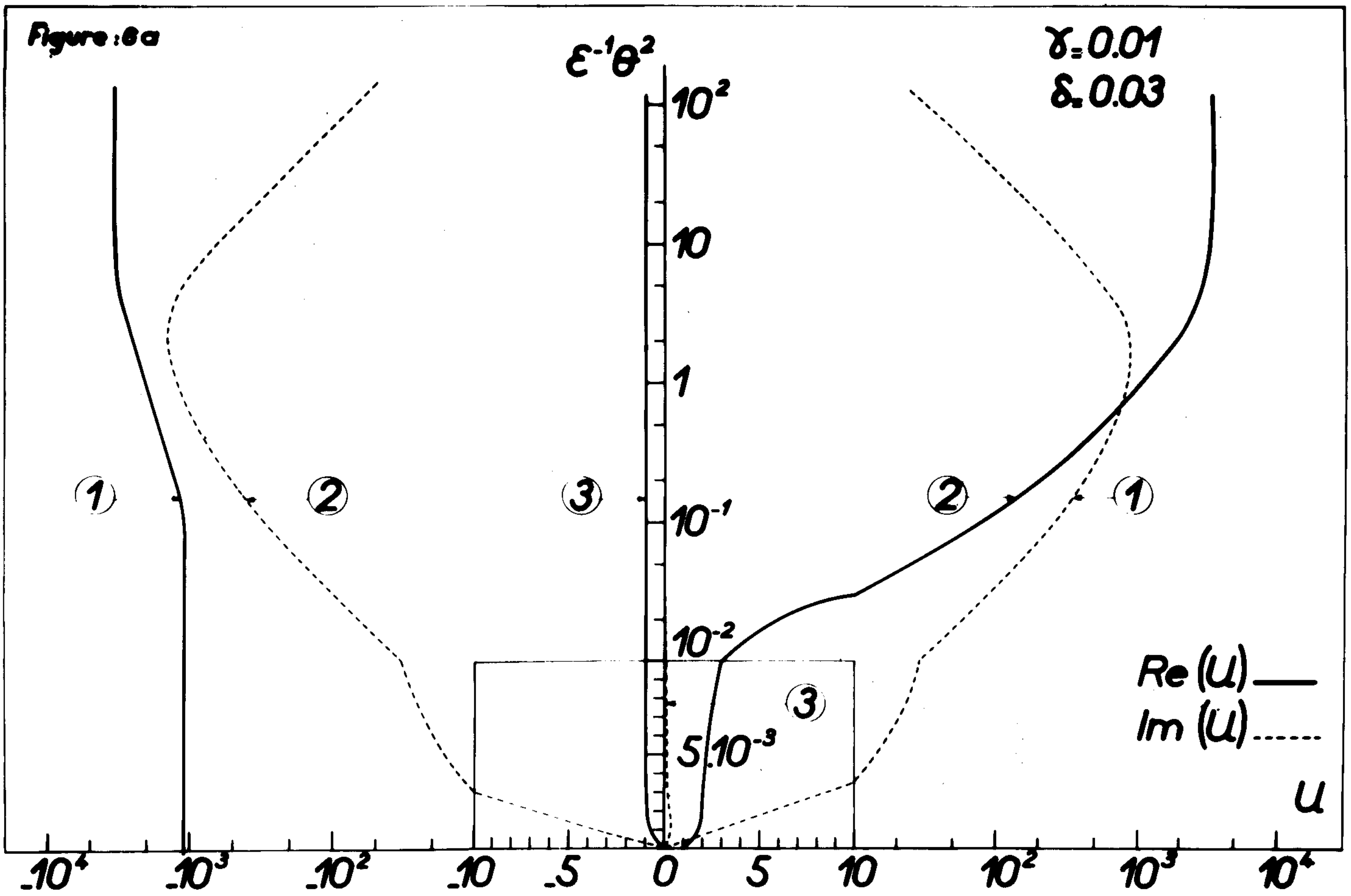


Figure : 6a
ε⁻¹θ²
γ=0.01
δ=0.03
10²
10
1
10⁻¹
10⁻²
5.10⁻³
1
2
3
2
1
3
Re(U)
Im(U)
U
-10⁴
-10³
-10²
-10
-5
0
5
10
10²
10³
10⁴

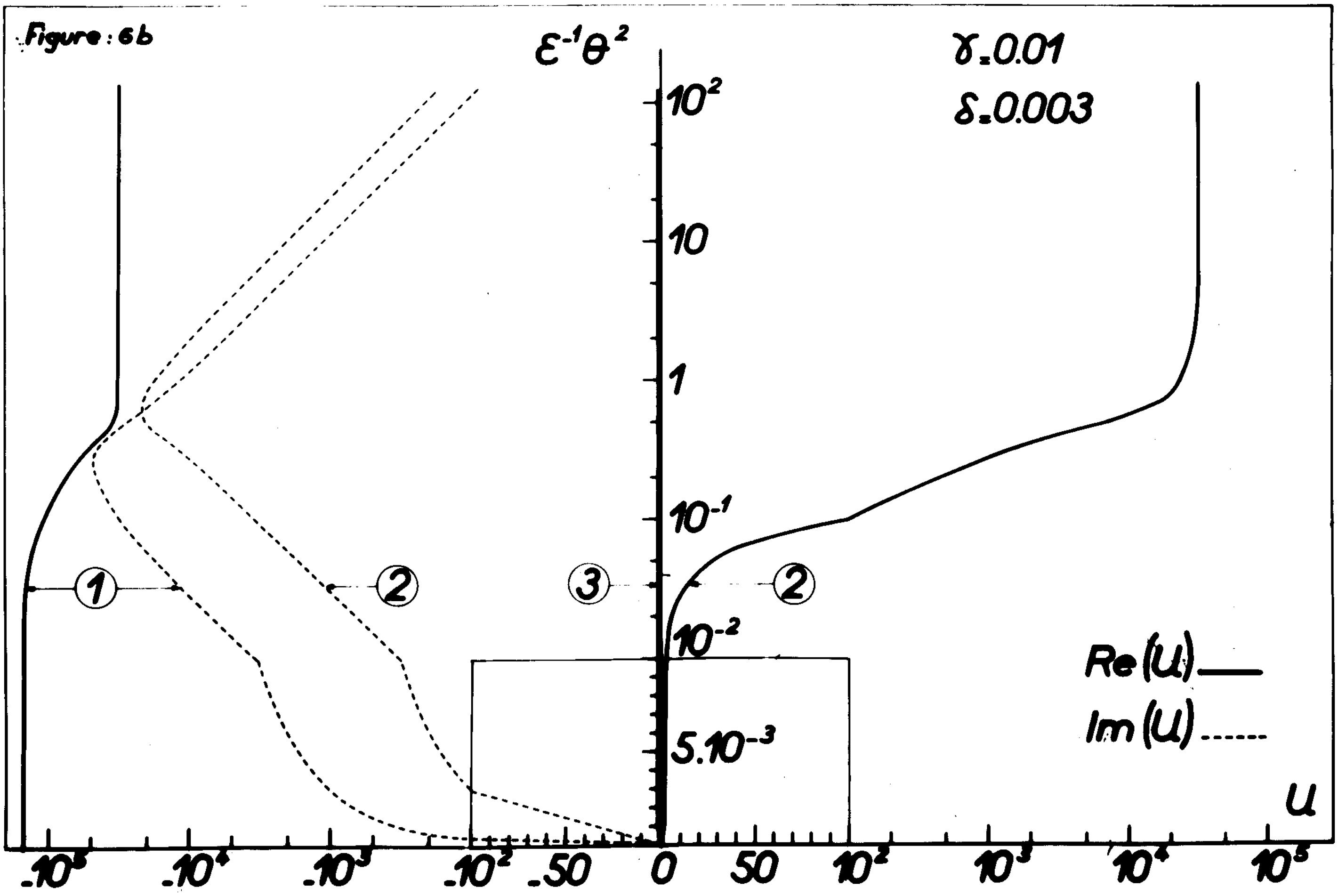


Figure : 6b
ε⁻¹θ²
γ=0.01
δ=0.003
10²
10
1
10⁻¹
10⁻²
5.10⁻³
1
2
3
2
Re(U)
Im(U)
U
-10⁵
-10⁴
-10³
-10²
-50
0
50
10²
10³
10⁴
10⁵