Effect of Sheaths on Drift Instabilities in Thermionic Plasmas

FRANCIS F. CHEN
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey
(Received 6 October 1964)

IN several recent experiments[1-3] in cesium and potassium plasmas, low-frequency oscillations propagating primarily in the azimuthal direction with the electron diamagnetic drift velocity have been found to be self-excited under certain conditions. In uniform magnetic fields, excitation requires the plasma to be ion-rich, so that ion sheaths exist near the cathodes. In curved magnetic fields, the oscillations are more easily excited and occur even in electron-rich plasmas. We believe that the criterion for excitation is determined by the conductivity of the sheaths: for electron sheaths, the conductivity is relatively high, and the plasma is stabilized at the metal end plates; for ion sheaths, the conductivity may be so low that the plasma does not *see* the end plates. Then a drift instability characterized by long axial wavelengths $\lambda_\parallel$ may grow even though $\lambda_\parallel$ is larger than the length $2L$ of the machine.

Consider a cylindrically symmetric, thermally ionized, isothermal plasma created between two hot cathodes in a strong magnetic field. Assume that perpendicular diffusion and volume recombination are negligible compared to losses at the end-plates[4]; then the parallel velocities v_{zi} and v_{ze} of the ion and electron fluids may be set equal to zero in equilibrium. The plasma potential V and density n are then constant ($V = 0$) in the body of the plasma but change abruptly in thin sheaths next to the cathodes ($V = V_c$). The sheath drop V_c determines the fluxes nv_i and nv_e of the two species in or out of the sheath, according to the sheath conditions given below. In equilibrium V_c is such that $nv_i = nv_e = 0$. When a perturbation V occurs in the plasma, the sheath drop $V - V_c$ will be locally changed, resulting in nonvanishing nv_i and nv_e along that line of force. We require that the values of $(nv_i)_L$ and $(nv_e)_L$ found from the perturbation analysis of the plasma region be consistent with the functions of $V - V_c$ given by the sheath conditions. We assume that the potential always varies monotonically from the cathode to the midplane, and that the frequency of oscillation ω is much smaller than the ion plasma frequency, so that the sheath changes adiabatically.

In a thermionic plasma, electrons lost to the cathodes are replenished by thermionic emission from the cathodes, and ions which recombine at the cathodes are reionized by contact ionization. The small fraction which is not reionized is replenished by ionization of a flux j_0 of neutral atoms impinging on the cathodes. To change from ion-rich to electron-rich plasmas, one may either decrease j_0 or increase j_T, the electron flux emitted at a temperature T according to Richardson's equation. In ion-rich plasmas ($V_c < 0$), the Coulomb barrier of the sheath reflects the plasma electrons trying to escape to the ends and accelerates the emitted flux j_T. In electron-rich ($V_c > 0$) plasmas, the opposite is true. Thus the sheath conditions at $z = \pm L$ are

$$V_c < 0: \quad \pm nv_e = nv_r \exp(eV_c/KT) - j_T \tag{1}$$

$$V_c > 0: \quad \pm nv_e = nv_r - j_T \exp(-eV_c/KT), \tag{2}$$

where $nv_r \equiv n(KT/2\pi m_e)^{\frac{1}{2}}$ is the random electron flux in the plasma. A similar but more complicated pair of equations obtains for the ions. However, we shall not need these, since the frequencies involved are so high and $\lambda_\parallel$ is so long that the ion parallel motions may be neglected during an oscillation. In equilibrium, V_c is such that $nv_e = 0$. We now replace n by $n_0 + n_1$ and V_c by $V_c - V_1$, where n_1 and V_1 are the perturbations in density and potential at the sheath edge, and linearize about the equilibrium. The resulting first-order boundary condition at $\pm L$ is

$$\pm(nv_{ez})_1 = \pm n_0 v_{ez} = n_0 v_r(\nu - \chi)\{\iota_T, 1\}, \tag{3}$$

where $\nu \equiv n_1/n_0$, $\chi \equiv eV_1/KT$, and $\iota_T \equiv j_T/n_0 v_r$. The quantities enclosed by $\{\ \}$ are for $V_c < 0$ and $V_c > 0$, respectively.

We must now specify the instability which is to satisfy this boundary condition. In the plasma under consideration there is neither an electric field nor a parallel current in equilibrium, and only the radial pressure gradient is available to drive the instability. It is well known that the "universal" instability of a collisionless plasma can occur under such circumstances. However, the effect depends on the existence of resonant particles, and we do not think that it occurs in the experiments,[1-3] where the collision frequency is larger than the oscillation frequency. Instead, the observed effect is more likely the overstability we reported earlier,[5,6] which depends on both finite resistivity η and finite $k_\parallel$. This overstability (wave 3 of Fig. 1 of Ref. 5) was obtained from a linearized treatment of the ion and electron fluid equations and the equations of continuity for perturbations of the form exp $(iky + ik_\parallel z - i\omega t)$ and for an exponential density profile in the x-direction. If we now restrict our attention to the drift waves ($\omega \ll \omega_c$) but allow the z-dependence to be arbitrary, the electron continuity equation corresponding to Eq. (44) of Ref. 6 becomes a differential equation in z:

$$\psi_e\nu - \delta\gamma(\nu - \chi) - i\epsilon^{-1}(\partial^2/\partial\zeta^2)(\nu - \chi) = 0, \quad (4)$$

where

$$\delta \equiv n_0^{-1}\,\partial n_0/\partial x, \qquad \gamma \equiv kr_L,$$

$$\zeta = z/r_L, \qquad \omega_c\psi_e = \omega - kv_e^{(0)},$$

$$\omega_c = eB/m_i, \qquad \epsilon \equiv en_0\eta/B, \qquad r_L \equiv (KT/m_i)^{\frac{1}{2}}/\omega_c.$$

The ion continuity equation does not involve z if ion parallel motions are neglected; it is therefore an algebraic relation between ν and χ, given by Eq. (31) of Ref. 6, and for $\psi^2 \ll 1$ can be written as follows:

$$\nu - \chi = \sigma\nu, \; \sigma = [(1 + 2\gamma^2)\psi + 2\delta\gamma]/(\gamma^2\psi + \delta\gamma), \quad (5)$$

where $\omega_c\psi \equiv \omega - kv_i^{(0)}$; $v_i^{(0)}$ and $v_e^{(0)} = -v_i^{(0)}$ are the zero-order diamagnetic drifts. Equation (5) can then be inserted in Eq. (4) to give an equation for ν. Since only ν depends on ζ, the symmetric solution is $\nu = a \cos \gamma_\parallel\zeta$, where $\gamma_\parallel$ is given by

$$\gamma_\parallel^2 = i\epsilon\sigma^{-1}(\psi_e - \delta\gamma\sigma). \quad (6)$$

We interpret $\gamma_\parallel$ to be the normalized propagation constant $k_\parallel r_L$ and require it to be real. Equation (6), when written out with the use of Eq. (5), is simply the dispersion relation, Eq. (18) of Ref. 5, for waves with arbitrary $\gamma_\parallel$. However, $\gamma_\parallel$ is now restricted to those eigenvalues satisfying the boundary condition (3). By Ohm's law, the first-order v_{ez} is approximately $v_{ez} = -\epsilon^{-1}(KT/m_i)^{\frac{1}{2}}(\partial/\partial\zeta)(\nu - \chi)$. Replacing $\nu - \chi$ by $\sigma\nu$, using the known form of ν, and equating v_{ez} to that given by Eq. (3) at $\zeta = \pm L/r_L$, we find the boundary condition

$$\gamma_\parallel \tan(\gamma_\parallel L/r_L) = \epsilon(2\pi m_e/m_i)^{-\frac{1}{2}}\{\iota_T, 1\}. \quad (7)$$

The right-hand side of this equation is fixed by the operating conditions of the machine. If the right-hand side is large, the argument of the tangent must be nearly an odd multiple of $\frac{1}{2}\pi$. The perturbation is then effectively tied at the ends, as if the sheaths did not exist; normally this implies such a large value of $\gamma_\parallel$ that drift instabilities cannot arise. If the right-hand side is small enough, the left-hand side is approximately $\gamma_\parallel^2 L/r_L$. Thus one can easily find the eigenvalue of $\gamma_\parallel$ for any given machine condition. One then looks at the dispersion relation for arbitrary $\gamma_\parallel$ to see if this value of $\gamma_\parallel$ can give an instability for the measured density gradient δ and a value of γ corresponding to a given azimuthal mode. This dispersion relation is given by Eqs. (6) and (5); we quote here the generalization of this to include field curvature and finite Larmor radius stabilization:

$$\tfrac{1}{2}\psi^2 + [\gamma(\tfrac{1}{2}\delta - \rho) + \tfrac{1}{2}iY]\psi - \delta(\rho - i\gamma Y) = 0, \quad (8)$$

where $\rho \equiv r_L/R$, $Y \equiv \epsilon^{-1}\gamma_\parallel^2/\gamma^2$, ψ is the normalized frequency in the frame moving with the ions, and R is the radius of curvature of B. For $Y \gg |\delta\gamma|$, $\rho = 0$, we have approximately

$$\psi \approx -2\delta\gamma + 2i\delta^2\gamma^2Y^{-1}. \quad (9)$$

If the field is slightly curved, there is an additional growth rate

$$\text{Im}\,\psi \approx -2\delta\rho Y^{-1}, \quad (10)$$

which is positive on the side where n_0 increases toward the center of curvature.

As a numerical example we have considered the lowest azimuthal mode of a 2300°K potassium plasma 3 cm in diam and 60 cm long, with $n_0 = 3 \times 10^{10}$ cm^{-3} and $B = 4$ kG. For electron sheaths, we use the "1" in the { } of Eq. (7); we then find $Y \approx 150$. The growth rate in Eq. (9) is so small for such a large value of Y that one would expect that the overstability would be damped. We have computed the damping due to electron diffusion and collisional ion viscosity; these effects turn out to be too weak to be important. The lower limit on the growth rate is probably set by the ion lifetime τ in the plasma; for $Y = 150$ the growth time is longer than the ion lifetime, and the perturbation cannot grow. For ion sheaths, we use the "ι_T" in Eq. (7), and we can ask the question the other way: If we require Im $\omega \gg 2\pi/\tau$, how small must ι_T be? Taking τ to be $\tau = N/(p\,dN/dt)$, where $N = 2Ln_0$, $dN/dt = \frac{1}{2}n_0\bar{v}$, and p = probability of recombining in a collision with the cathode, we find $\tau = 20$ msec for $p = 0.05$. Then we find that ι_T must be less than 3.8×10^{-4}. Since $\iota_T = \exp(eV_c/KT)$, this implies $|eV_c| > 1.6$ V for the oscillations to be produced. This is in reasonable agreement with observations.[1] If B is curved, Eqs. (9) and (10) show that r_L/R must be greater than $1.5\,\delta\gamma^2$ before the destabilization due to the curvature becomes noticeable. Published data[2] are not sufficiently detailed to check this point.

Physically, it is clear that with large ion sheaths only a small fraction of the plasma electrons can reach the cathode through the sheath barrier, and therefore the sheath is an effective insulator. Perturbations with $\lambda_\parallel \gg L$ can exist because E_θ does not have to vanish at the sheath edge although it does vanish at the cathode surface. $\nabla \times \mathbf{E} = 0$ is maintained by a difference in $E_\parallel$ in the sheath at different azimuths. The study of drift instabilities is basic to the understanding of anomalous diffusion of plasma across a magnetic field. The end-plate stabilization discussed here makes cesium and potassium plasmas valuable for such studies because an initially stable plasma can be obtained and the onset of drift instabilities can be observed. Details on these calculations will be published.

---

[1] N. D'Angelo and R. W. Motley, Phys. Fluids **6**, 422 (1963).

[2] N. D'Angelo, D. Eckhart, G. Grieger, E. Guilino, and M. Hashmi, Phys. Letters **11**, 525 (1963).

[3] H. Lashinsky, Phys. Rev. Letters **12**, 121 (1964); **13**, 47 (1964).

[4] S. v. Goeler, Phys. Fluids **7**, 463 (1964).

[5] F. F. Chen, Phys. Fluids **7**, 949 (1964).

[6] F. F. Chen, Princeton University, Plasma Physics Laboratory Report Matt-227 (1963).