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Corrections

DSP THEORY FOR EXPERIMENTALISTS

by

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A. Introduction

Our purpose here is to elucidate the physical processes involved in charge neutralization and wave propagation in the Dawson isotope separation process using the two-ion hybrid resonance. It is well known that for $k_{||}/k_{\perp} << (m/M)^{1/2}$ a resonance occurs at the lower hybrid frequency $\omega_L \approx (\omega_c \Omega_c)^{1/2}$, while for $k_{||}/k_{\perp} >> (m/M)^{1/2}$ the plasma supports an electrostatic ion cyclotron wave obeying $\omega^2 \approx \Omega_c^2 + k_{\perp}^2 c_s^2$. For $k_{||} > k_{\perp}$, the dominant effect should be an ordinary ion acoustic wave. How does the two-ion hybrid fit into this scheme, and how does its frequency change with the angle of propagation?

To answer these questions, we start with the simplest set of equations that still contain all the important effects of linear wave propagation. The long list of simplifications will be removed one by one in subsequent papers as more complicated phenomena are treated.

B. Simplifying Assumptions and their Implications

1. Unbounded, homogeneous plasma. Homogeneity implies the absence of drift-wave or Kelvin-Helmholtz effects arising from zero-order drifts. Unboundedness in space and time means that the excitation process is not considered; i.e., the finite transit time and excitation time of the resonating particles are neglected. Less importantly, the eigenmode structure of the waves within the vacuum vessel is also neglected.

2. Fluid ions and electrons. The use of fluid equations without a viscosity term implies the neglect of two types of kinetic effects. The first is, of course, Landau and cyclotron damping. This occurs whenever ω or $\omega \pm n\omega_c$ is close to $k_{||} v_{thermal}$ for either ions or electrons. Incorporation of these effects in a homogeneous plasma is straightforward and involves adding the plasma dispersion function (Z-function). In an inhomogeneous plasma, inverse Landau damping could lead to drift-wave excitation. The second effect is finite Larmor radius. For the bulk plasma, this effect is small, but for the spun-up particles this effect is dominant. Thus, we are omitting the effect of the accelerated particles on the wave.

3. Collisionless plasma. This assumption is made only to simplify the algebra. Collisions with neutrals can be included easily by the replacement of ω with $\omega+i\nu$ in appropriate places. Coulomb collisions are more tedious to incorporate, particularly in an inhomogeneous plasma, since the cross section is density dependent. The main effect on the bulk plasma will be to broaden resonances and shift their frequencies slightly. In an inhomogeneous plasma there will also be a tendency toward self-excitation due to the resistive drift wave effect. For the accelerated particles, of course, collisions can be a dominant effect in determining their energy.

4. Electrostatic waves. The neglect of electromagnetic terms is justified if the phase velocity is less than the Alfvén speed ($\omega/k_{||} >> c_A$), as is the case in DSP experiments. The theory, however, extends to regions where this inequality is violated; the dispersion curves will be shown dashed there.

5. Quasineutrality. We assume $n_e = \Sigma Z_i n_i$ rather than use Poisson's equation. This results in an error of order $k^2\lambda_D^2 << 1$. Including this term is trivial but unnecessarily complicates the algebra.

6. Cold ions ($T_i = 0$). This simplification has three effects, all of which are unimportant. First, T_i increases c_s slightly in a plasma with $T_e >> T_i$. Second, parallel T_i causes ion Landau damping, which is important only for $k_{||}/k_\perp$ larger than occurs in experiments. Third, perpendicular T_i causes finite Larmor radius effects, which should be very small for the bulk plasma.

7. Plane waves. We assume infinite plane waves. Cylindrical geometry complicates the algebra but does not introduce new physical effects. Also, plane waves comprise both right and left circular polarizations, so cyclotron resonances are included.

8. Low frequencies. We neglect electron cyclotron effects by assuming $\omega << \omega_c$.

C. Linear dispersion relation

1. Equation of motion for ions (linearized)

$$M n_o \frac{\partial \underline{v}}{\partial t} = Z n_o e \left(\underline{E} + \underline{v} x \underline{B}_o\right) \quad (1)$$

For $\underline{v} \propto e^{-i\omega t}$, this has the solution

$$v_x = \frac{iZe}{M\omega}\left(E_x + \frac{i\Omega}{\omega}E_y\right)\left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1}$$

$$v_y = \frac{iZe}{M\omega}\left(E_y - \frac{i\Omega}{\omega}E_x\right)\left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1} \tag{2}$$

$$v_z = \frac{iZe}{M\omega}\; E_z$$

where $$\Omega \equiv \frac{ZeB_o}{M}, \quad \underline{B}_o = B_o\,\hat{z} \tag{3}$$

2. Equation of motion for electrons (linearized)

$$mn_o \frac{\partial \underline{v}}{\partial t} = -en_o\left(\underline{E} + \underline{v} \times \underline{B}_o\right) - KT\underline{\nabla}n, \tag{4}$$

where $T = T_e$ ($T_i = 0$), and the electrons are taken to be isothermal. The electrostatic approximation allows us to replace $\underline{E}$ by $-\nabla\phi$ and to combine the $\underline{E}$ term with the $\underline{\nabla}n$ term. Defining

$$\psi \equiv \phi - \frac{kT}{e}\frac{n}{n_o}, \tag{5}$$

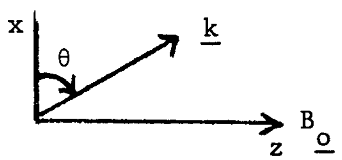
we see from analogy to (2) that the solution to (4) is

$$\begin{aligned} v_x &= \frac{ie}{m\omega}\left(\nabla_x\psi - \frac{i\omega_c}{\omega}\nabla_y\psi\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} \\ v_y &= \frac{ie}{m\omega}\left(\nabla_y\psi + \frac{i\omega_c}{\omega}\nabla_x\psi\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}. \\ v_z &= \frac{ie}{m\omega}\nabla_z\psi \end{aligned} \tag{6}$$

3. Plane waves. We now assume $\psi \propto \exp(i\underline{k}\cdot\underline{r})$ and, without loss of generality, take the x-axis to lie along $\underline{E}$. Thus,

$$\underline{\nabla} \rightarrow i\underline{k}, \qquad k_y = 0, \qquad k_x,\ k_z \neq 0, \text{ and}$$

$$\frac{k_z}{k_x} = \tan\theta \equiv T$$



Eqs. (2) for the ions then become

$$v_x = k_x \frac{\Omega}{\omega} \frac{\phi}{B_o} \left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1}$$

$$v_y = -ik_x \frac{\Omega^2}{\omega^2} \frac{\phi}{B_o} \left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1} \tag{7}$$

$$v_z = k_z \frac{\Omega}{\omega} \frac{\phi}{B_o}$$

As for the electrons, we may neglect 1 compared to ω_c^2/ω^2 in Eq. (6), but we must retain the first-order terms in ω/ω_c (the polarization drifts) because otherwise we would lose the lower hybrid resonance. Eqs. (6) then become

$$v_x \simeq \frac{ek_x}{m} \frac{\omega}{\omega_c^2} \psi$$

$$v_y \simeq \frac{iek_x}{m\,\omega_c} \psi \tag{8}$$

$$v_z = \frac{-ek_z}{m\omega} \psi$$

4. <u>Electron continuity equation</u> (linearized)

$$\frac{\partial n}{\partial t} + n_o \left(\underline{\nabla} \cdot \underline{v}\right) = 0, \tag{9}$$

$$\therefore \quad \frac{n}{n_o} = \frac{1}{\omega} \left(k_x v_x + k_z v_z\right) \qquad \left(k_y = 0\right) \tag{10}$$

Using (8), we have

$$\frac{n}{n_o} = \frac{k_x v_x}{\omega} \left(1 - T^2 \frac{\omega_c^2}{\omega^2}\right), \text{ so that}$$

$$\psi = \phi - \frac{KT}{e} \frac{k_x v_x}{\omega} \left(1 - T^2 \frac{\omega_c^2}{\omega^2}\right). \tag{11}$$

This can now be used in Eqs. (8) to solve for $\underline{v}$ in terms of ϕ, with n eliminated. The result is

$$v_x = k_x \frac{\omega}{\omega_c} \frac{s}{B_o} \phi$$

$$v_y = ik_x \frac{s}{B_o} \phi \tag{12}$$

$$v_z = -k_z \frac{\omega_c}{\omega} \frac{s}{B_o} \phi,$$

where

$$s \equiv \left[1 + \frac{KT}{m}\left(\frac{k_x^2}{\omega_c^2} - \frac{k_z^2}{\omega^2}\right)\right]^{-1} . \tag{13}$$

Note that ψ, as defined in (5), is the deviation from a Boltzmann electron distribution. The latter would give

$$n_o + n = n_o e^{e\phi/KT} \quad \therefore \quad n = n_o \frac{e\phi}{KT} , \; \psi = 0 . \tag{14}$$

The velocity components, proportional to ψ/m, would then be in the indeterminate form 0/0, since (14) is valid only if $m \to 0$. We keep the effect of electron inertia in v_z so that we can see what happens at very small k_{11}. This gives v_z a finite value in (12c). Although the deviation from Boltzmann arises from inertia in the parallel motion, the perpendicular components v_x and v_y are also affected; in fact, they are also proportional to ψ. This is because the E and ∇n terms in the equation of motion normally cancel, and only the deviation from Boltzmann gives a net force from E and ∇p. Finally, note that if $T \to 0$, s becomes 1, and Eqs. (12) reduce to the usual formulas for the polarization drift (v_x), the $\underline{\text{ExB}}$ drift (v_y), and acceleration due to E_{11} (v_z).

We now substitute Eqs. (12) back into Eq. (10):

$$\frac{n}{n_o} = \left(\frac{k_x^2}{\omega_c B_o} - \frac{k_z^2 \omega_c}{\omega^2 B_o}\right) s\phi = \frac{\omega_c \phi}{B_o} \frac{\dfrac{k_x^2}{\omega_c^2} - \dfrac{k_z^2}{\omega^2}}{1 + \dfrac{kT}{m}\left(\dfrac{k_x^2}{\omega_c^2} - \dfrac{k_z^2}{\omega^2}\right)}$$

Remembering that $n = n_e$ here, we finally obtain

$$\frac{n_e}{n_o} = \frac{e}{m} \phi \left[v_{th}^2 + \left(\frac{k_x^2}{\omega_c^2} - \frac{k_z^2}{\omega^2} \right)^{-1} \right]^{-1} , \qquad (15)$$

where

$$v_{th} \equiv \left(KT/m\right)^{1/2} \qquad (16)$$

5. Ion continuity equation (linearized). Eq. (10) also holds for each ion species j. Substituting (7) into (10), we obtain

$$\frac{n_j}{n_{oj}} = \frac{\Omega_j \phi}{\omega^2 B_o} \left[k_z^2 + k_x^2 \left(1 - \frac{\Omega_j^2}{\omega^2} \right)^{-1} \right]. \qquad (17)$$

6. Poisson's equation. For two ion species (major species 1, minor species 2), we have

$$\nabla \cdot E = 4\pi e \left(Z_1 n_1 + Z_2 n_2 - n_e \right) \simeq 0. \qquad (18)$$

The r.h.s. is set equal to 0 according to the quasineutrality assumption discussed earlier. For the unperturbed densities, we have

$$Z_1 n_{o1} + Z_2 n_{o2} = n_{oe}, \text{ or } Z_1 \alpha_1 + Z_2 \alpha_2 = 1, \qquad (19)$$

where $\alpha_j \equiv n_{oj}/n_{oe}$ (20)

Substituting (15) and (17) into (18) for the perturbed densities, we obtain the dispersion relation:

$$n_{o1} \frac{Z_1 e\phi}{M_1 \omega^2} \left[k_z^2 + k_x^2 \left(1 - \frac{\Omega_1^2}{\omega^2} \right)^{-1} \right] + n_{o2} \frac{Z_2 e\phi}{M_2 \omega^2} \left[k_z^2 + \right.$$

$$\left. + k_x^2 \left(1 - \frac{\Omega_2^2}{\omega^2} \right)^{-1} \right] = n_{oe} \frac{e\phi}{m} \left[\frac{KT}{m} + \left(\frac{k_x^2}{\omega_c^2} - \frac{k_z^2}{\omega^2} \right)^{-1} \right]^{-1} . \qquad (21)$$

Defining the sound velocities

$$c_{sj} \equiv \left(Z_j \; KT/M_j\right)^{1/2}, \tag{22}$$

we can write this as

$$\frac{\alpha_1 c_{s1}^2}{\omega^2}\underbrace{\left[k_z^2 + k_x^2\left(1 - \frac{\Omega_1^2}{\omega^2}\right)^{-1}\right]}_{(A)} + \frac{\alpha_2 c_{s2}^2}{\omega^2}\underbrace{\left[k_z^2 + k_x^2\left(1 - \frac{\Omega_2^2}{\omega^2}\right)^{-1}\right]}_{(B)} =$$

$$\underbrace{\left[1 + \left(\frac{k_x^2 \; v_{th}^2}{\omega_c^2} - \frac{k_z^2 \; v_{th}^2}{\omega^2}\right)^{-1}\right]^{-1}}_{(C)} . \tag{23}$$

The space change effects appear explicitly here: Term A is that due to the major species, term B that due to the minor species, and term C that due to the electrons. Eq. (23) can be written

$$\left[1 + \left(\frac{k_x^2 \; v_{th}^2}{\omega_c^2} - \frac{k_z^2 \; v_{th}^2}{\omega^2}\right)^{-1}\right]\left[\alpha_1 \; c_{s1}^2\left(\frac{k_x^2}{\omega^2 - \Omega_1^2} + \frac{k_z^2}{\omega^2}\right) + \alpha_2 c_{s2}^2\left(\frac{k_x^2}{\omega^2 - \Omega_2^2} + \frac{k_z^2}{\omega^2}\right)\right] = 1 . \tag{24}$$

D. Limiting cases

1. Strictly perpendicular propagation ($k_z = 0$). In this limit, Eq. (24) becomes

$$\left(1 + \frac{\omega_c^2}{k^2 v_{th}^2}\right)\left(\frac{\alpha_1 k^2 c_{s1}^2}{\omega^2 - \Omega_1^2} + \frac{\alpha_2 k^2 c_{s2}^2}{\omega^2 - \Omega_2^2}\right) = 1 . \tag{25}$$

The "1" on the left can safely be neglected, since the condition $k^2 v_{th}^2 << \omega_c^2$ must be fulfilled anyway, if cyclotron damping is to be ignored. Since

$$\frac{\omega_c^2}{k^2 v_{th}^2} \; \frac{k^2 c_{s1}^2}{\omega^2 - \Omega_1^2} = \frac{\omega_c^2 \left(KT/M_1\right) \; Z_1}{\left(KT/m\right) \; \left(\omega^2 - \Omega_1^2\right)} = \frac{\omega_c \Omega_1}{\omega^2 - \Omega_1^2} ,$$

we have

$$\underset{(A)}{\frac{\alpha_1\Omega_1}{\omega^2 - \Omega_1^2}} + \underset{(B)}{\frac{\alpha_2\Omega_2}{\omega^2 - \Omega_2^2}} = \underset{(C)}{\frac{1}{\omega_c}} . \tag{26}$$

a) Lower hybrid. This quadratic in ω^2 yields two roots. We assume $\alpha_2 << \alpha_1 \simeq 1$, since "2" indicates the minor species. The upper root is not greatly affected by the α_2 term, since ω will turn out nowhere near Ω_2; hence we neglect term B and obtain

$$\omega^2 = \Omega_1^2 + \omega_c\Omega_1 \simeq \omega_c\Omega_1 . \tag{27}$$

This is the lower hybrid frequency of the major species. The term in Ω_{pi} which normally appears in the lower hybrid is missing here because we did not use the full Poisson equation. It is trivial to include it if one wishes.

b) Two-ion hybrid. The lower root of the quadratic can be approximated as follows. If ω is sufficiently close to Ω_2, term B in Eq. (26) can be larger than ω_c^{-1} even for small α_2. Neglecting term C in Eq. (26) is tantamount to neglecting electron space change and requiring the two ion species to cancel each other's space charge. It is clear from (26) that this can occur if ω lies between Ω_1 and Ω_2. Setting the sum of A and B in (26) equal to 0, we obtain

$$\alpha_1\Omega_1\left(\omega^2 - \Omega_2^2\right) = -\alpha_2\Omega_2\left(\omega^2 - \Omega_1^2\right)$$

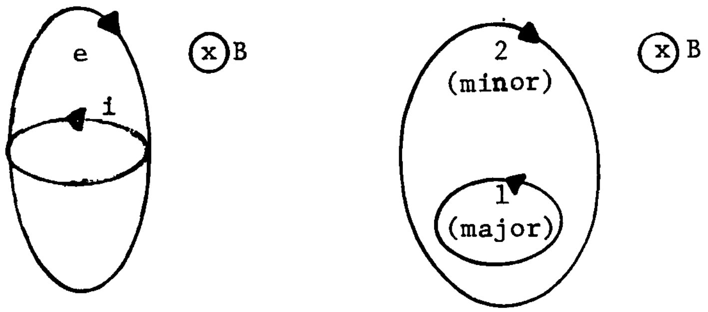
$$\omega^2\left(\alpha_1\Omega_1 + \alpha_2\Omega_2\right) = \alpha_2\Omega_1^2\,\Omega_2 + \alpha_1\Omega_2^2\,\Omega_1$$

$$\omega^2 = \Omega_1\Omega_2\,\frac{\alpha_1\Omega_2 + \alpha_2\Omega_1}{\alpha_1\Omega_1 + \alpha_2\Omega_2} . \tag{28}$$

This is the Buchsbaum two-ion resonance frequency, which clearly approaches Ω_2 as α_2 becomes small. Eq. (26) shows clearly that the smaller α_2 is, the closer ω must be to Ω_2 in order for term B to dominate over term C. Thus, the resonance is sharper for lower α_2, and the minor species is preferentially excited in a self-adjusting manner. Of course, in this particular case $(k_z = 0)$, the electron term is small, arising only from the

the polarization drift across B_o. When electron flow along B_o is allowed, the electron contribution to space change will be much larger, and the 2-ion resonance correspondingly sharper.

c. Physical intepretation. These two oscillations may be pictured as follows. In the lower hybrid, the major ion species has a velocity given by Eq. 7, while the electrons have a velocity given by Eq. (12). If $\omega = (\omega_c\Omega_1)^{1/2}$ is substituted into v_{xi} and v_{xe}, it is seen that they are nearly the same. Since all the gradients are in the x direction, this means that these two fluids move together and have no change separation. Looking at the y components, we see that v_{ye} is ω_c/ω times larger than v_{xe}; hence, the electron orbits are elongated in the y direction. The ions have v_{yi} smaller than v_{xi} by a factor Ω_1/ω, so the opposite is true for the ions, as shown in (a) below.



(a) (b)

The signs of the v_y terms yields the gyration directions shown. These happen to be the same as for the cyclotron motions, even though the Larmor gyrations have been neglected in this simple picture. The electron orbit is the result of a polarization drift in the x direction and an ExB drift in the y direction. (The polarization drift is the motion in the direction of $\underline{E}$ which occurs in the first half-cycle of a cyclotron gyration after $\underline{E}$ is suddenly switched on or changes sign.) The ion orbit results from an equal polarization drift in the x direction and a reduced ExB drift in the y direction, reduced because $\Omega_c \ll \omega$.

In the 2-ion hybrid, let $\Omega_2 > \Omega_1$ for definiteness. The frequency (28) results from setting the v_x's in Eq. (7a) equal and opposite the each other for the two species when multiplied by n_{oj}. This can occur only if $\Omega_1 < \omega < \Omega_2$. The sign of the v_y term is -i for species 1 and +i for species 2, showing that the heavier species gyrates in the normal direction, while the lighter species goes oppositely. Since v_y is Ω/ω times as large as v_x, the orbit is slightly elongated in the x direction for species 1 and in the y direction for species 2. This is shown in figure (b) above. The orbits are not equal in size because the velocities must be weighted by α_j so that the ion currents in the x direction are equal and opposite.

2. Propagation at "large" angles θ. Consider the electron term C in Eq. (23). For $k_z/k_x >> m/M$, the $(k_z v_{th}/\omega)^2$ term dominates over the $(k_x v_{th}/\omega_c)^2$ term when ω is in the ion cyclotron range. Furthermore, if $(\omega/k_z v_{th})^2 << 1$, the k_z term can be neglected, and the term C reduces to 1. This is the limit of Boltzmann electrons.

Consider now the terms A and B. If $k_z^2 >> k_x^2 \left(1 - \frac{\Omega^2}{\omega^2}\right)^{-1}$,

the dispersion relation simply becomes

$$\omega^2 = k_z^2 \left(\alpha_1 c_{s1}^2 + \alpha_2 c_{s2}^2\right), \tag{29}$$

which is an ion acoustic wave with c_s^2 weighted by the relative abundances. On the other hand, there is a large range of k_z where it is small enough to be neglected in A and B but large enough to make term C equal to 1. In this range, Eq. (23) becomes

$$\alpha_1 k_x^2 c_{s1}^2 \left(\omega^2 - \Omega_2^2\right) + \alpha_2 k_x^2 c_{s2}^2 \left(\omega^2 - \Omega_1^2\right) = \left(\omega^2 - \Omega_1^2\right)\left(\omega^2 - \Omega_2^2\right). \tag{30}$$

This is a quadratic in ω^2 with two roots.

a. Electrostatic ion cyclotron wave. The first root can be found by neglecting the minor species 2. We then have

$$\omega^2 = \Omega_1^2 + k_x^2 \left(KT_e/M_1\right). \tag{31}$$

Introducing the α_2 term as a perturbation to this solution, we find

$$\omega^2 = \Omega_1^2 + \alpha_1 k_x^2 c_{s1}^2 \left(1 + \frac{\alpha_2 k_x^2 c_{s2}^2}{\alpha_1 k_x^2 c_{s1}^2 + \Omega_1^2 - \Omega_2^2}\right). \tag{32}$$

This is the electrostatic ion cyclotron wave in the presence of two ion species.

b. Minor species resonance. The second root must have $\omega \simeq \Omega_2$, so that the α_2 term can become comparable to the others. Taking $\omega = \Omega_2 + \varepsilon$ and expanding in ε, we easily obtain

$$\omega^2 = \Omega_2^2 + \alpha_2 k_x^2 c_{s2}^2 \frac{\Omega_1^2 - \Omega_2^2}{\alpha_1 k_x^2 c_{s1}^2 + \Omega_1^2 - \Omega_2^2} . \tag{33}$$

This is a cyclotron resonance of the minor species, with a frequency shift different from that of Eq. (28) for the 2-ion hybrid. Here the minor species space charge cancels the sum of the electron space change and the major species space charge. If the major species did not participate, the first term in Eq. (30) would be dropped, and the fraction in Eq. (33) would be unity. If the electrons did not participate, the r.h.s. of Eq. (30) would be dropped, and we would recover the 2-ion hybrid, Eq. (28). Thus, a wave with $\omega^2 = \Omega_2^2 + 0(\alpha_2)$ exists for both "large" and small k_z. These lie, however, on different branches of the dispersion curve, as the following computation shows.

E. Computed dispersion curves

1. Dimensionless form. Denoting the major species by "1" and the minor species by "2", we define the following quantities normalized to species 1:

$$T \equiv \tan\phi = k_z/k_x, \quad \alpha_1 \equiv \frac{n_{o1}}{n_{oe}}, \quad \alpha_2 = \left(1 - Z_1\alpha_1\right)/Z_2$$

$$\Omega \equiv \omega/\Omega_1, \quad R \equiv \Omega_2/\Omega_1 = \left(M_1/M_2\right)\left(Z_2/Z_1\right)$$

$$\frac{c_{s2}^2}{c_{s1}^2} = \frac{Z_2 KT}{M_2} \cdot \frac{M_1}{Z_1 KT} = \left(\frac{M_1}{M_2}\right)\left(\frac{Z_2}{Z_1}\right) = R \tag{34}$$

$$\kappa \equiv k_x c_{s1}/\Omega_1, \quad \varepsilon \equiv \left(\frac{m}{M_1}\right) \quad Z_1 = \frac{\Omega_1}{\omega_c}$$

$$v_{th}^2 = \frac{KT}{M}\frac{M}{m} = c_{s1}^2/\varepsilon .$$

The dispersion relation (24) can be rewritten

$$\left[1 + \left(\kappa^2 \varepsilon - \frac{\kappa^2 T^2}{\varepsilon\Omega^2}\right)^{-1}\right] \left[\alpha_1 \kappa^2 \left(\frac{1}{\Omega^2 - 1} + \frac{T^2}{\Omega^2}\right) + \right.$$

$$\left. \alpha_2 \kappa^2 R \left(\frac{1}{\Omega^2 - R^2} + \frac{T^2}{\Omega^2}\right)\right] = 1,$$

or

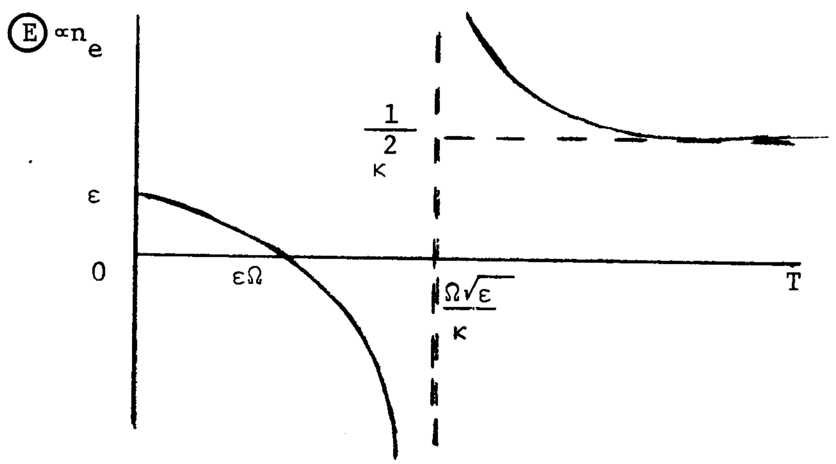
$$\alpha_1 \left(\frac{1}{\Omega^2 - 1} + \frac{T^2}{\Omega^2}\right) + \alpha_2 R \left(\frac{1}{\Omega^2 - R^2} + \frac{T^2}{\Omega^2}\right) = \frac{1}{\kappa^2 + \dfrac{\varepsilon\Omega^2}{\varepsilon^2\Omega^2 - T^2}} \tag{35}$$

(A) (B) (C) (D) (E)

n_1 n_2 n_e

The three terms correspond to the first-order n_1, n_2, and n_e, as indicated.

2. Behavior of the electron term. We are interested in how the nature of the roots Ω changes with T (i.e., with k_z for given k_x). If Ω were fixed, the term E representing the electron space charge would vary as follows as T is changed.



a) When T is 0, the n_e term is small ($\sim\varepsilon$) because the wavefronts are parallel to B_o, and the only way the electrons can move at low frequency is via their small polarization drifts. There are two roots. If term A in Eq. (35) balances term E, we get the lower hybrid frequency $\Omega^2 \simeq \varepsilon^{-1}$. If term A balances term C, we get the 2-ion hybrid $\Omega^2 \simeq R^2$.

b) As T is increased, one first arrives at a point where $T = \varepsilon\Omega$, and $n_e = 0$. This happens at very small k_z such that the electron motion parallel to B_o cancels the effect of electron motion across B_o. Clearly, the lower hybrid cannot exist here. Since $\Omega \simeq \varepsilon^{-1/2}$ for the lower hybrid, this occurs at $T = \varepsilon/\varepsilon^{1/2} = \varepsilon^{1/2}$; hence the well-known result that the lower hybrid cannot exist for $|k_z/k_x| > (m/M)^{1/2}$. On the other hand, the 2-ion hybrid persists ($T \simeq \varepsilon R$ for it), because the electrons are not involved in this resonance anyway.

c) As T is further increased, the electron term is approximately

$$\frac{1}{\kappa^2 - \frac{\varepsilon\Omega^2}{T^2}} .$$

This goes to $\pm\infty$ at $T \simeq \Omega\sqrt{\varepsilon}/\kappa$. To balance the large electron density, one or the other of the ion terms must be large; this can happen only near a cyclotron resonance. Thus, one would expect roots near $\Omega = 1$ (major resonance) and $\Omega = R$ (minor, or 2-ion, resonance). However, the condition $T = \Omega\sqrt{\varepsilon}/\kappa$ corresponds, in either case, to $\omega/k_z = v_{th}$, which implies strong electron Landau damping. Thus, both resonances are inadequately described by fluid theory in this region. In fact, it is just the resonance of electron parallel motion with ion cyclotron motion that caused n_e to diverge in the first place. This effect is non-physical, because all electrons do not have a common velocity v_{th}.

d) For larger T, the electron term reduces to κ^{-2}, independent of m/M. This is the limit of Boltzmann electrons. Terms B and D, representing ion acoustic effects, can still be neglected if $T \ll 1$. When term A balances term E, we have the electrostatic ion cyclotron wave. When term A balances term C, we have the two-ion hybrid.

e) For large T, term B eventually dominates the ion motion except near the cyclotron resonances. If the electron term is κ^{-2}, terms B and E yield $\omega = k_z c_{s1}$, the ion acoustic wave of the major species. If Ω is sufficiently close to R, term C can balance term B, with E being a small correction. This yields

$$\frac{\alpha_1 T^2}{\Omega^2} + \frac{\alpha_2 R}{\Omega^2 - R^2} = 0,$$

$$\Omega^2 = R^2 \frac{\alpha_1 T^2}{\alpha_1 T^2 + \alpha_2 R} \simeq R^2, \quad \omega^2 \simeq \Omega_2{}^2 .$$

Thus, the minor resonance survives even at large T. The same is true for the major resonance, because for $T^2 >> \Omega^2$ terms A and B can cancel each other, with E as a correction term. We then get

$$\omega^2 = \Omega_1{}^2 \left(1 + \frac{k_x{}^2}{k_z{}^2}\right)^{-1} .$$

The frequency must lie below Ω_1 in this case. For both these cyclotron waves, one must have

$$k_z < \Omega_c / v_{thi}$$

to avoid ion Landau damping.

Because of its insensitivity to electron motions, the minor resonance has the remarkable property that it exists over almost the whole range of k_z/k_x.

3. Computation. For illustrative purposes, we have computed the dispersion relation (35) for parameters appropriate to M1A:

Argon $M_1 = 40\ M_H$ $M_2 = 36\ M_H$

$Z_1 = 1$ $Z_2 = 1$

$\alpha_1 = .9964$ $\alpha_2 = .0036$

$KT_e = 2\text{eV}$, $KT_i = 0$

$B_o = 3$ kG, $n_o = 10^{11}$ cm^{-3}

$R = 10/9$, $\varepsilon = 1.362 \times 10^{-5}$

$k_x = m/r = 8/4$ cm^{-1} (this corresponds to an m = 8 excitation of a 16-segment plate)

$\Omega_1 = 7.186 \times 10^5$ sec^{-1}

$c_{s_1} = 2.19 \times 10^5$ cm/sec

$\kappa^2 = 0.371$ (fixed, while T is varied)

Eq. (35) can be solved for $\Omega(T)$ on an HP-25 calculator by iteration. Such root tracing, however, incurs troubles when the curves cross. Alternatively, Eq. (35) can be solved directly for $T(\Omega)$, since it is a quadratic in T^2. Unfortunately, the equation is too long for the 49-step program of the HP-25, and furthermore there is a problem of underflow when subtracting two large numbers in finding the lower root. The following overlapping approximations, however, give accurate results.

a) Far from cyclotron resonances. The quadratic for T^2 has the solution

$$\frac{T^2}{\Omega^2} = \frac{B}{2A}\left[1 \pm \left(1 - \frac{4AC}{B}\right)^{1/2}\right], \quad \text{where} \tag{36}$$

$$A \equiv \kappa^2 (\alpha_1 + \alpha_2 R)$$

$$B \equiv 1 + \left(\varepsilon + \kappa^2\varepsilon^2\right)\left(\alpha_1 + \alpha_2 R\right) - \kappa^2\left(\frac{\alpha_1}{\Omega^2 - 1} + \frac{\alpha_2 R}{\Omega^2 - R^2}\right)$$

$$C \equiv \varepsilon^2 - \left(\varepsilon + \kappa^2\varepsilon^2\right)\left(\frac{\alpha_1}{\Omega^2 - 1} + \frac{\alpha_2 R}{\Omega^2 - R^2}\right).$$

If the denominators in C are not resonant, C is always small, and the square root can be expanded. We then obtain

$$T_+^2 = \Omega^2 \left(\frac{B}{A} - \frac{C}{B}\right) \tag{37}$$

$$T_-^2 = \Omega^2 \frac{C}{B} . \tag{38}$$

b) Near the cyclotron resonances

Small T ($T \ll 1$): In this case, since $\Omega^2 \simeq 1$, the acoustic terms B and D in Eq. (35) can be neglected, and we have

$$\left(\kappa^2 + \frac{\varepsilon}{\varepsilon^2 - \frac{T^2}{\Omega^2}}\right)\left(\frac{\alpha_1}{\Omega^2 - 1} + \frac{\alpha_2 R}{\Omega^2 - R^2}\right) = 1 . \tag{39}$$

Defining

$$Q \equiv \frac{\alpha_1}{\Omega^2 - 1} + \frac{\alpha_2 R}{\Omega^2 - R^2} \tag{40}$$

we have the solution

$$\frac{T_1^2}{\Omega^2} = \varepsilon \left(\varepsilon + \frac{1}{\kappa^2 - Q^{-1}}\right) . \tag{41}$$

Large T ($T \gg \varepsilon^{1/2}$): Here we retain the acoustic terms but can simplify the electron term E in (35) to κ^{-2}. Thus, Eq. (35) becomes

$$\kappa^2 \left[\frac{\alpha_1}{\Omega^2 - 1} + \frac{\alpha_2 R}{\Omega^2 - R^2} + \frac{T^2}{\Omega^2} \left(\alpha_1 + \alpha_2 R\right)\right] = 1, \tag{42}$$

whose solution is

$$\frac{T_2^2}{\Omega^2} = \frac{\kappa^{-2} - Q}{\alpha_1 + \alpha_2 R} \text{, with Q as in (40).} \tag{43}$$

4. <u>Results.</u> Fig. 1 shows the frequency $\Omega = \omega/\Omega_1$ vs. $T = k_z/k_x$ over 10 orders of magnitude in T. Regions where kinetic or electromagnetic effects are large are indicated by the dotted lines. It is seen that most of the curves lie clear of these regions. Cyclotron damping has not been shown, but the same is true for that. In particular, these effects are unimportant in the region of experimental interest, shown by the circle.

There are three branches representing essentially the waves in the major species alone, plus a straight line representing the two-ion or minor species resonance. The upper branch starts at the lower hybrid frequency ($\Omega = 271$), and at $T \simeq \varepsilon^{1/2}$ changes to a modified electron cyclotron wave

$$\omega^2 \simeq \omega_c^2 \tan^2 \theta. \tag{44}$$

This frequency is the result of the vanishing of the electron denominator ($\varepsilon^2 = T^2/\Omega^2$) in Eq. (39). This means that at this frequency the electron density perturbation can be made arbitrarily small through a cancellation of the x and z motions of electrons, and therefore a small ion space charge can be neutralized regardless of what it is exactly. If the term of order ω^2/ω_c^2 had been kept in the electron equation of motion, we would probably have obtained the more familiar result $\omega^2 = \omega_c^2 \sin^2 \theta$ for the cyclotron wave in a magnetic field. In any case, this branch is too high in frequency to be of interest here.

The middle branch starts, at small k_z, as a lower hybrid oscillation reduced in frequency by the finite angle:

$$\omega^2 \simeq \omega_c \Omega_c \tan^2 \theta. \tag{45}$$

At $k_z/k_x \simeq (m/M)^{1/2}$, it changes to the electrostatic ion cyclotron wave, seen as the flat part of the curve extending over two orders of magnitude in k_z/k_x. At $T \simeq 1$, this curve bends upwards into a 45° line representing the ion acoustic wave.

The lowest branch of Fig. 1 starts as the continuation of the ion wave $\omega^2 = k_z^2 c_{s1}^2$; at $\Omega \simeq 1$ it changes to an ion cyclotron wave $\omega \lesssim \Omega_{c1}$. This branch must always stay below Ω_{c1}, as can be seen from the dispersion relation (35). In this range, the minor species term is negligible because $\alpha_2 << \alpha_1$

and $\omega \neq \Omega_2$, while the electron term is $\kappa^{-2} = 0(1)$ (Maxwellian electrons). Thus, the two ion terms A and B must cancel each other: the ion charge separations due to parallel and perpendicular motions are equal and opposite. Since $T^2/\Omega^2 >> 1$ in this range, $\Omega^2 - 1$ must be small and negative.

So far, the waves we have described involve the major species only. The minor species is so dilute that its effects are felt only near the line drawn at $\Omega = R$. The fine structure near the minor resonance can seen by expanding the scale, as in Figs. 2 and 3. The horizontal scales are the same in these two graphs, but the vertical scale is successively expanded. It is seen how one branch goes from the two-ion hybrid into the electrostatic ion cyclotron mode, and the other branch goes from the frequency-reduced lower hybrid into the minor species cyclotron resonance. The frequency shift from Ω_2 is slightly different on either side of the intersection. The region of the major resonance has not been shown on an enlarged scale because there is no fine structure to be seen there.

5. Further results. We have also computed a case where $\Omega_{c2} < \Omega_{c1}$; namely, a U^{238} minor species in an A^{40} major:

$M_1 = 40\ M_H$	$M_2 = 238\ M_H$
$Z_1 = 1$	$Z_2 = 1$
$\alpha_1 = 0.9977$	$\alpha_2 = .0033$
$KT_e = 2$ eV	$KT_i = 0$
$B_o = 15$ kG	$n_o = 3 \times 10^{11}\ cm^{-3}$
$R = 0.168$	$\varepsilon = 1.362 \times 10^{-5}$
$k_x = 8/(5\ cm) = 1.6\ cm^{-1}$	
$\Omega_1 = 3.59 \times 10^6\ sec^{-1}$	$\Omega_2 = 6.04 \times 10^5\ sec^{-1}$
$c_{s1} = 2.19 \times 10^5$ cm/sec	$c_{s2} = 8.97 \times 10^4$ cm/sec
$\kappa^2 = 9.53 \times 10^{-3}$	

The resulting dispersion curves are shown on Fig. 4. The minor species resonance is seen to extend over all values of k_z/k_x with undiscernible frequency shift. The electrostatic ion cyclotron wave in this case lies far from the minor resonance. Figures 5 and 6 show the structure near Ω_{c2} on successively larger scale so that the frequency shift can be seen. Figure 7 shows the structure near the major resonance Ω_{c1}. Note that excitation of the major resonance could be achieved if either branch near Ω_{c1} were excited. The right-hand branch, with $\omega \lesssim \Omega_{c1}$, has a phase velocity lying well within the ion velocity distribution and would be Landau damped. But the electrostatic ion cyclotron branch in this case should be observable; and, since its frequency is within 0.5% of Ω_{c1}, it should be capable of accelerating argon ions for over 100 cyclotron periods. The relative difficulty in detecting a major resonance with a radial energy analyser may be due to the fact that the wave saturates before the ion orbits become large. For the minor species, since there are few resonant particles, their orbits can become large even if the amplitude of the wave near Ω_{c2} is small.

Finally, we have computed the case of a 50-50% D-T mixture, in which the two ion species play equal roles. Our previous approximations [Eqs. (37), (38), (41), and (43)] are still valid, since they did not depend on the smallness of α_2. Taking tokamak parameters, we have:

$M_1 = 2\ M_H$	$M_2 = 3\ M_H$
$Z_1 = 1$	$Z_2 = 1$
$\alpha_1 = 0.5$	$\alpha_2 = 0.5$
$KT_e = 1$ keV	$KT_i = 0$
$B_o = 30$ kG	$n_o = 3 \times 10^{13}\ \text{cm}^{-3}$
$R = 2/3$	$\varepsilon = 2.723 \times 10^{-4}$
$k_x = 8/(4\ \text{cm}) = 0.2\ \text{cm}^{-1}$	
$\Omega_1 = 1.437 \times 10^8\ \text{sec}^{-1}$	$\Omega_2 = 9.58 \times 10^7\ \text{sec}^{-1}$
$c_{s1} = 2.19 \times 10^7$ cm/sec	$c_{s2} = 1.79 \times 10^7$ cm/sec
$\kappa^2 = 9.28 \times 10^{-4}$	

The result is shown in Fig. 8. Because neither ion species plays a minor role, the wave frequencies do not lie close to either cyclotron frequency. The two-ion resonance occurs at the geometric mean gyrofrequency, and there are two electrostatic ion cyclotron waves. The two cyclotron waves for $k_z > k_x$ are Landau damped. The two isotopes cannot be separated when their concentrations are comparable.

F. The Space Charge Problem

Although this simple treatment was not designed to address the problem of space charge, much of the physics is already clear. As long as the plasma is unbounded, it can provide space charge neutralization for waves near the minor resonance. If $k_z/k_x < (m/M)^{1/2}$, the major species moves so as to cancel the space charge of the minor species. If $k_z/k_x > (m/M)^{1/2}$, the electrons move back and forth along the magnetic field to cancel this space charge. Which mode will occur in practice depends on the sheath bounding the plasma axially. In either case, a wave near Ω_2 can be propagated from an electrostatic exciter and will cyclotron-accelerate the minor species.

If the plasma is bounded, and the spun-up species goes outside the plasma, space charge cancellation will not occur naturally. In a plasma with a gentle radial density gradient, space charge problems will occur only outside a radius where the main plasma does not contain a sufficient reservoir of charge (major ions in the small k_z case and electrons in the intermediate k_z case). The collector, therefore, must extend into a plasma of finite density.

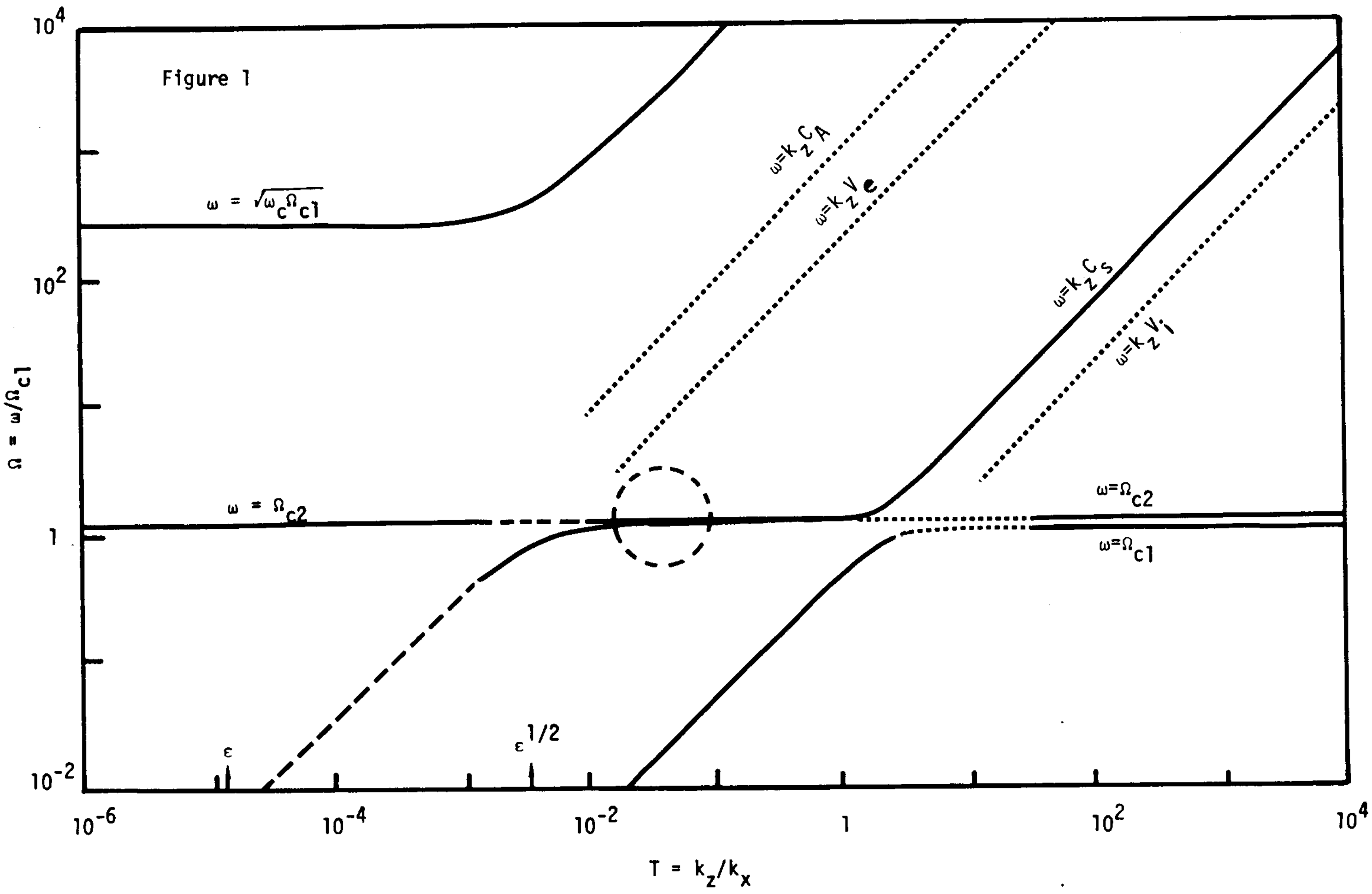


Figure 1
ω = √(ω_c Ω_c1)
ω=k_z C_A
ω=k_z V_e
ω=k_z C_s
ω=k_z V_i
ω = Ω_c2
ω=Ω_c2
ω=Ω_c1
ε
ε^1/2
10^4
10^2
1
10^-2
10^-6
10^-4
10^-2
1
10^2
10^4
Ω = ω/Ω_c1
T = k_z/k_x

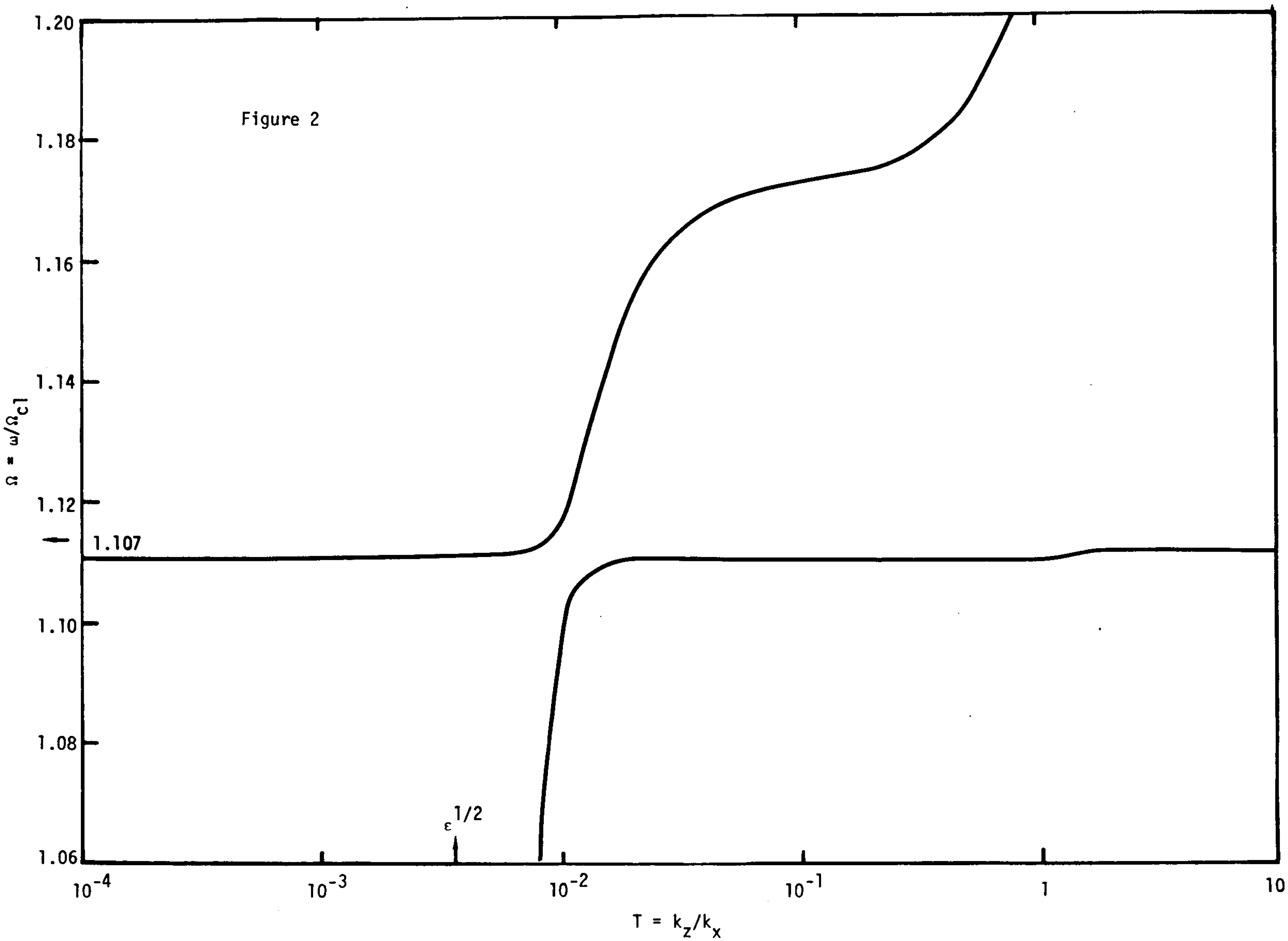

1.20
1.18
1.16
1.14
1.12
1.107
1.10
1.08
1.06
Ω = ω/Ω_c1
ε^1/2
10^-4
10^-3
10^-2
10^-1
1
10
T = k_z/k_x

Figure 2

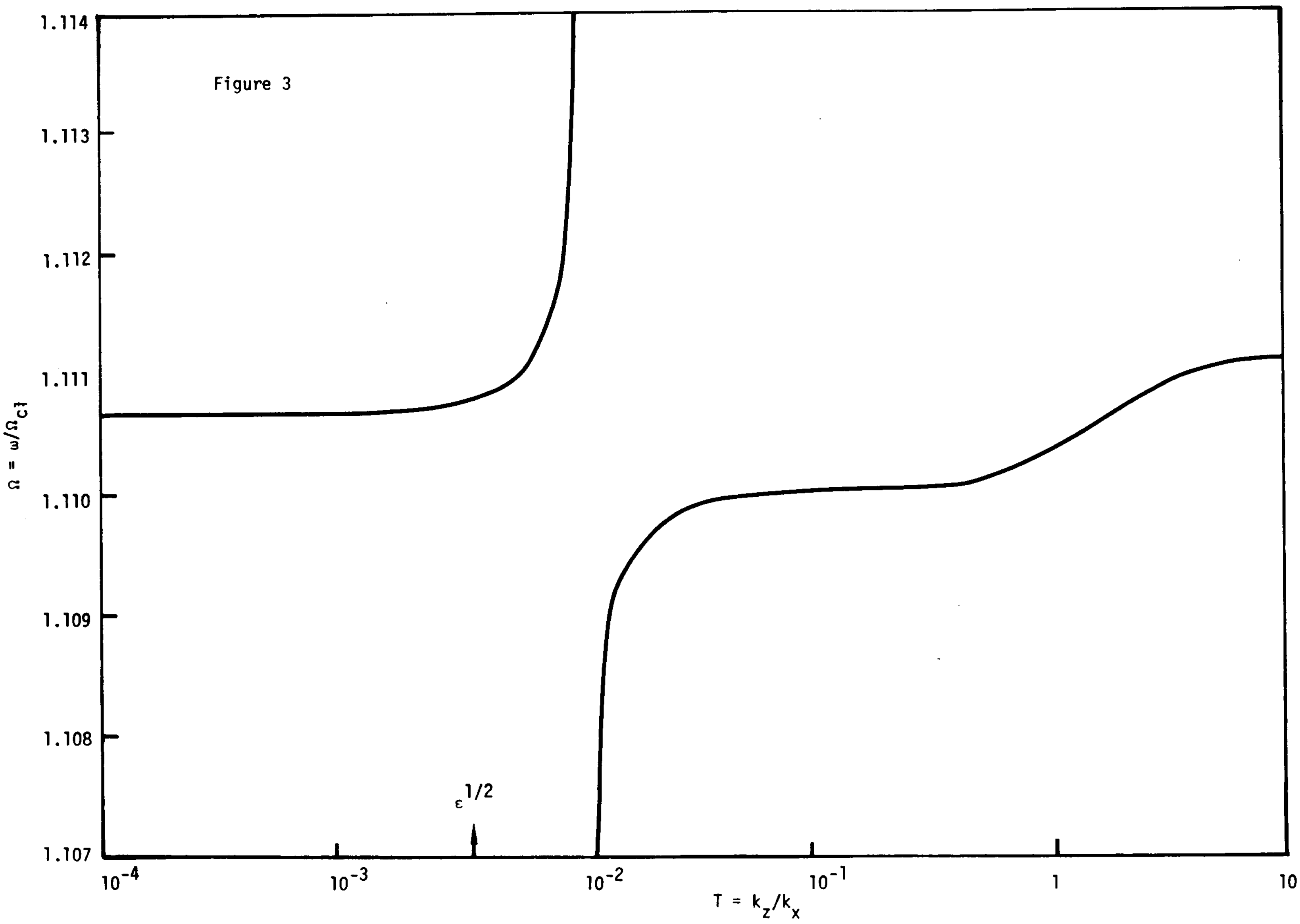

1.114
1.113
1.112
1.111
1.110
1.109
1.108
1.107
$\Omega = \omega/\Omega_C$
$\varepsilon^{1/2}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10
$T = k_z/k_x$

Figure 3

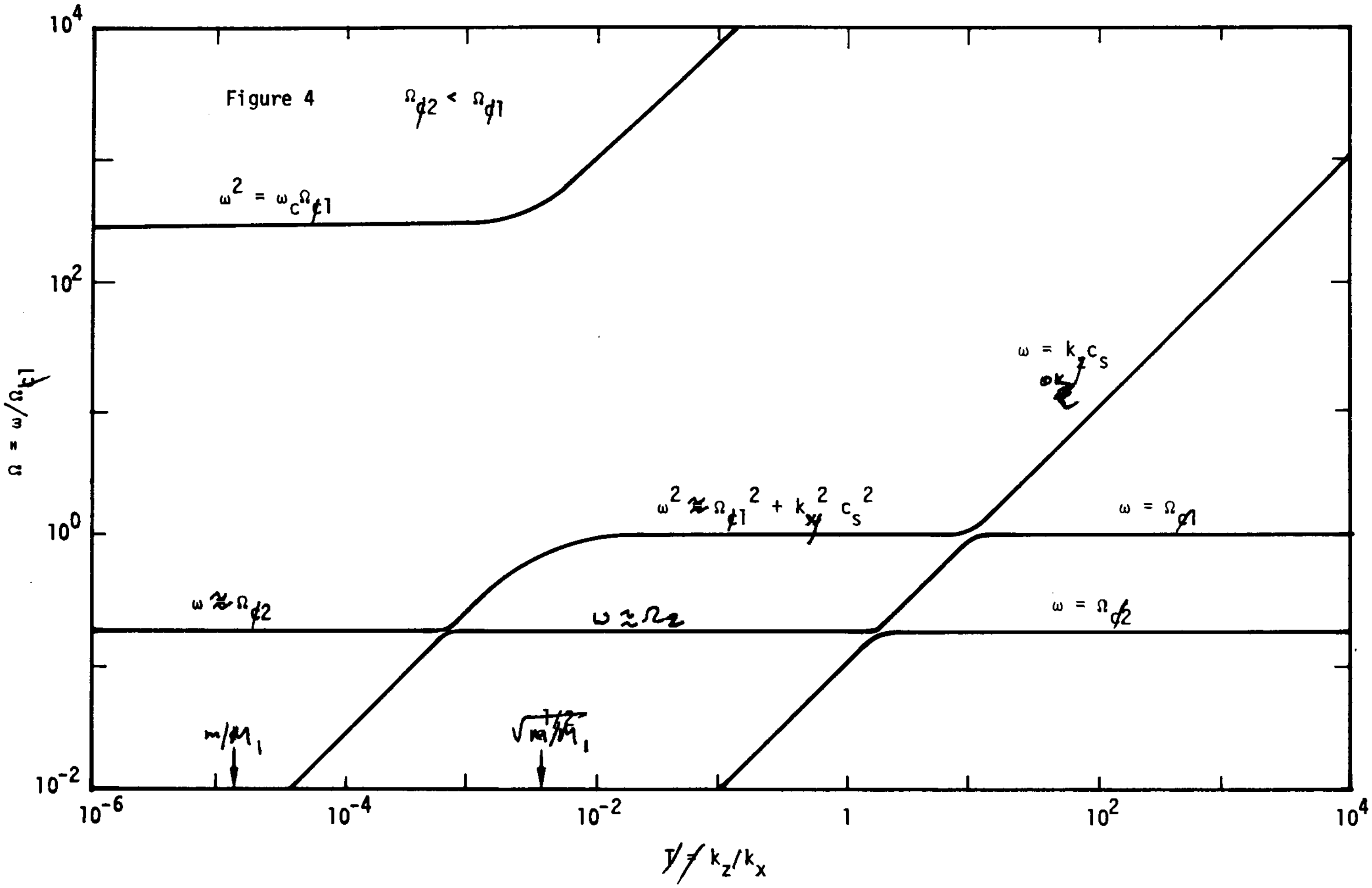


Figure 4
Ω_d2 < Ω_d1
ω² = ω_c Ω_d1
ω = k_z c_s
ω² ≈ Ω_d1² + k_x² c_s²
ω = Ω_d1
ω ≈ Ω_d2
ω ≈ Ω_2
ω = Ω_d2
m/M_1
√(m/M_1)
10^4
10^2
10^0
10^-2
10^-6
10^-4
10^-2
1
10^2
10^4
Ω = ω/Ω_1
T = k_z/k_x

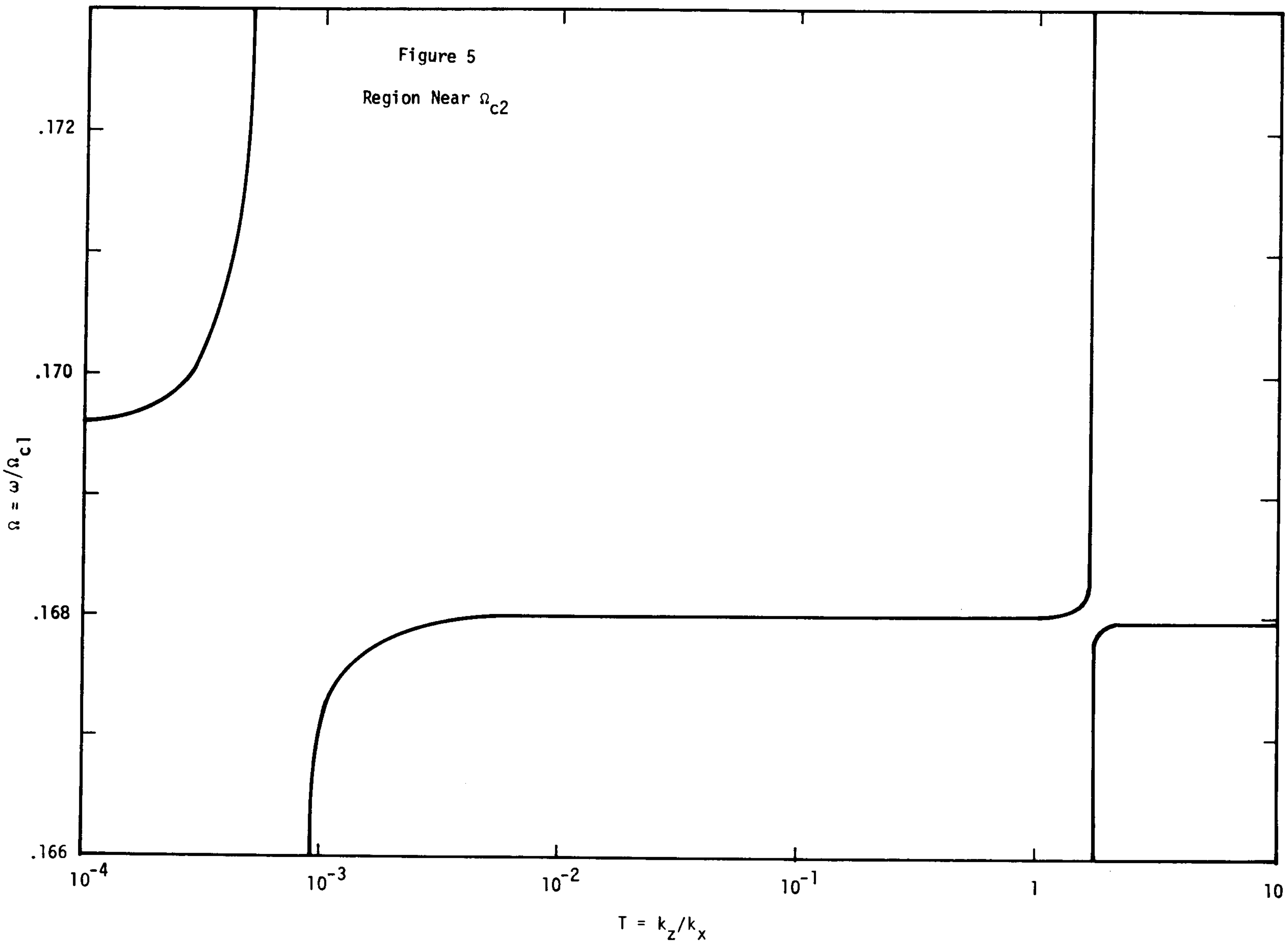


Figure 5

Region Near Ω_{c2}

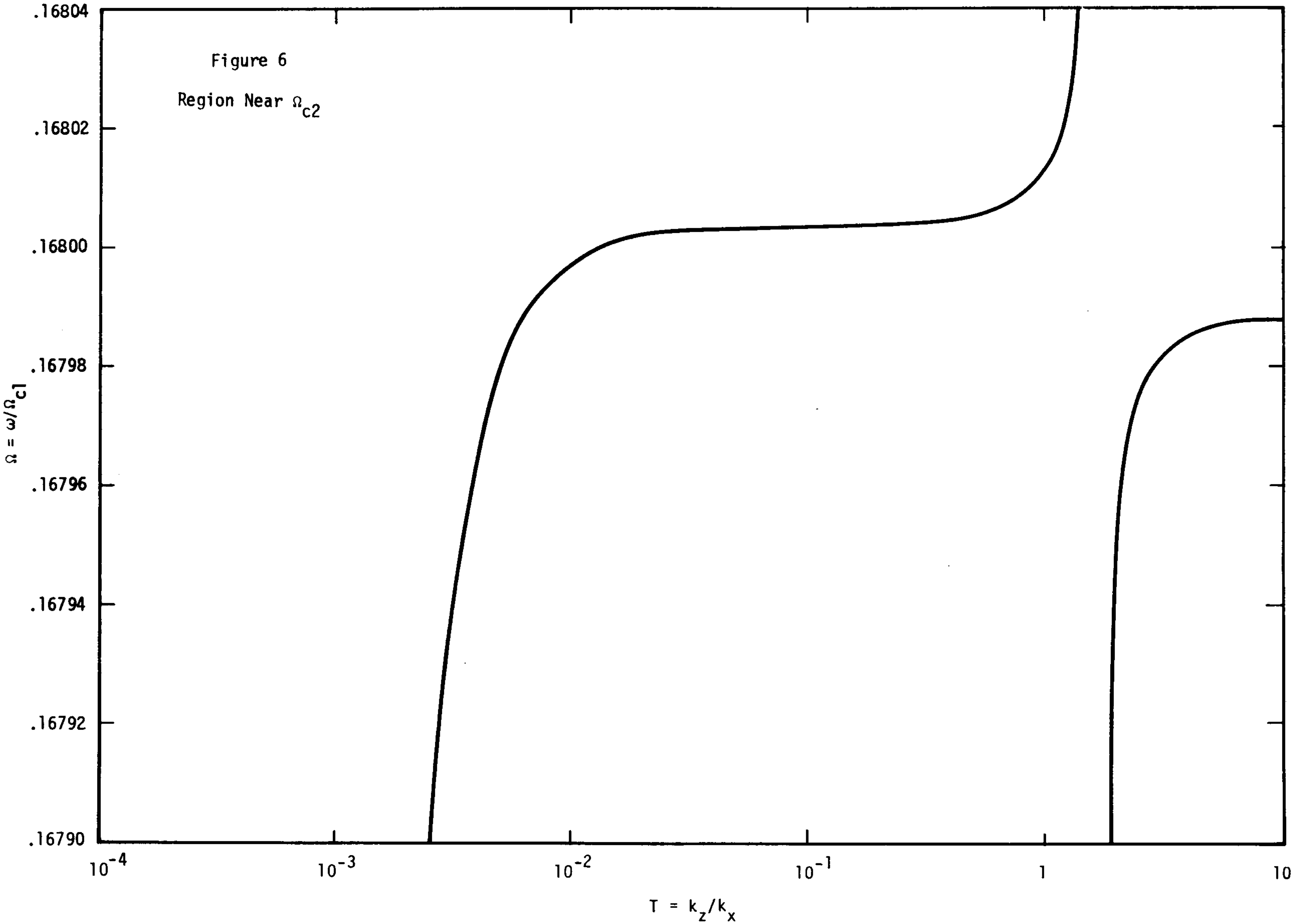


Figure 6

Region Near Ω_{c2}

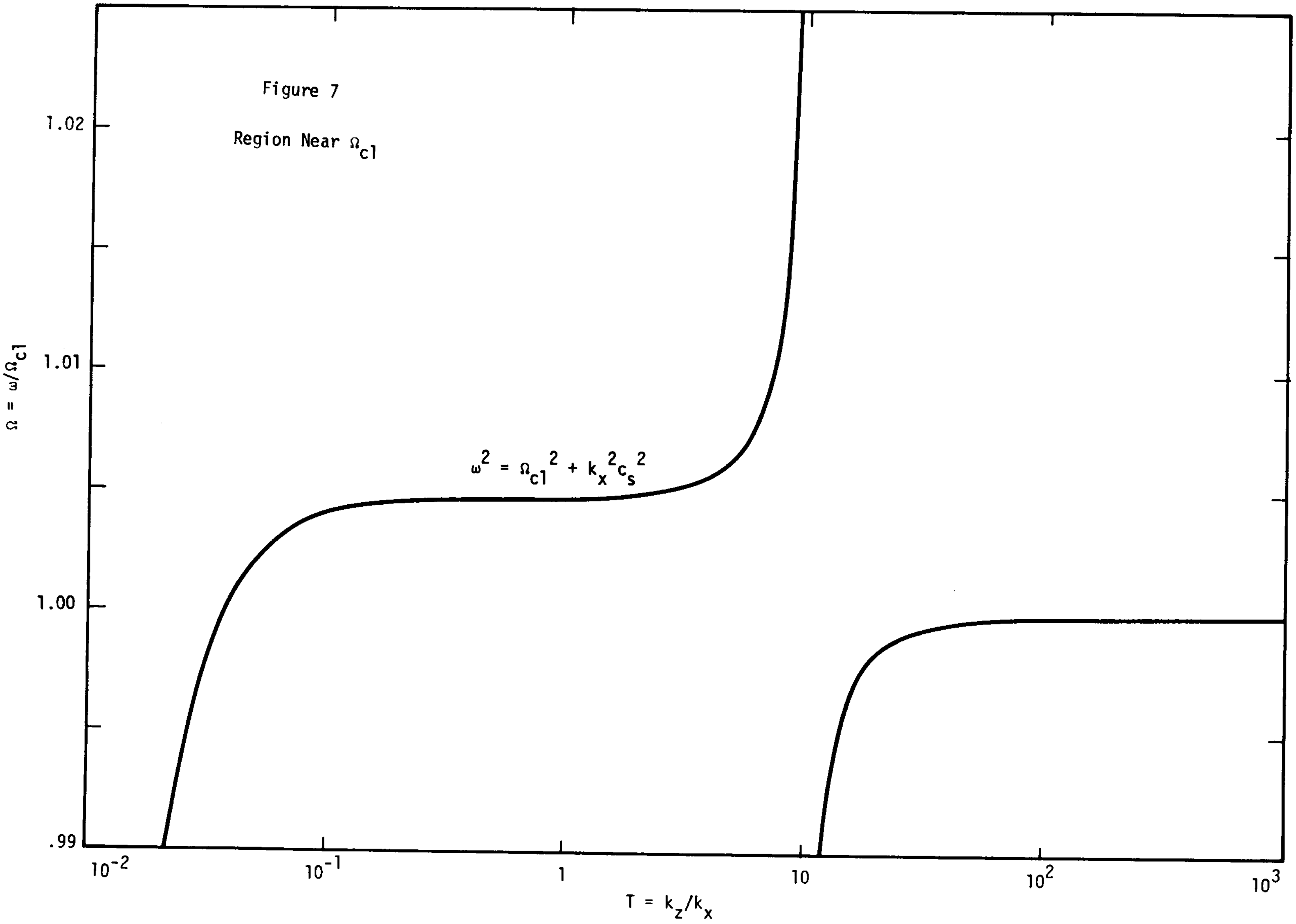


Figure 7

Region Near Ω_{c1}

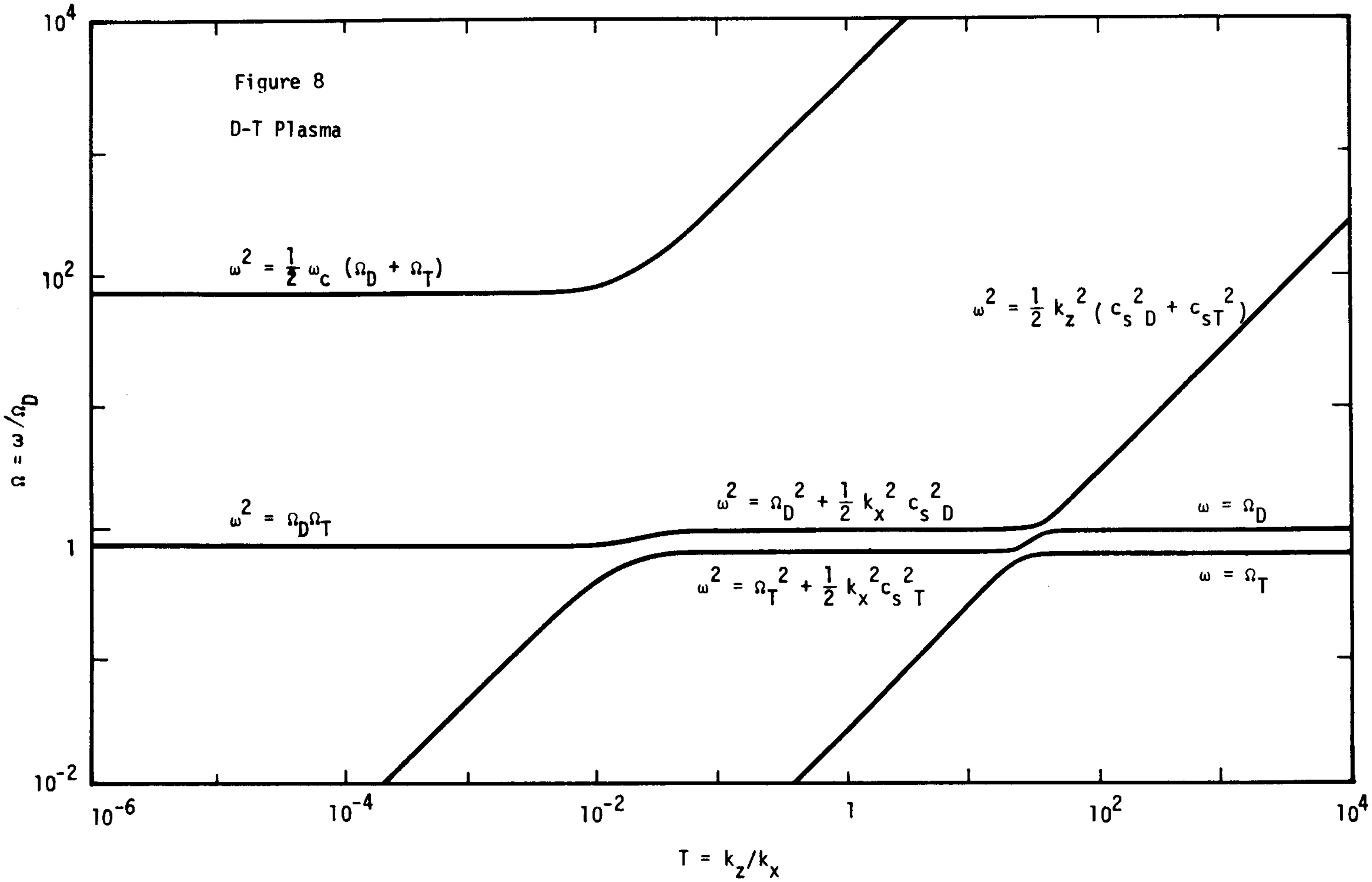


Figure 8

D-T Plasma