THICKNESS OF COMBINED BOHM-LANGMUIR SHEATHS

Francis F. Chen

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L. Higgins

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When a large voltage is applied to a plane probe in a plasma, a sheath of sizable thickness compared to a Debye length λ_D can be created. In the case of cylindrical or spherical probes, the perturbation caused by the probe falls sufficiently fast with distance that an exact solution can be obtained; this has been done by H. Lam using aerodynamic boundary layer analysis techniques[1]. In the case of a plane probe in a magnetic field, however, the perturbation caused by the probe extends to infinity, and the problem is not well defined. We present here a practical but non-rigorous analysis and indicate the approximations involved.

Consider a plane probe of unit area biased to a potential -V relative to the plasma potential. We assume the ions to be cold ($T_i << T_e$) and the electrons to be in a Maxwellian distribution at temperature T. If the density far from the probe is n_o, a normal Debye sheath has a thickness $d_1 \simeq 2\,\lambda_D$, where

$$\lambda_D = (\varepsilon_o KT/n_o e^2)^{1/2} \qquad \text{(mks)} \tag{1}$$

If the probe is biased at a negative voltage much larger than KT, there is an additional sheath next to the probe, where the electron density can be neglected and the local potential takes the large drop to the probe potential. The thickness d_2 of this region is given by the Child-Langmuir formula

$$d_2 = \frac{2}{3}\left(\frac{2e}{M}\right)^{1/4}\left(\frac{\varepsilon_o}{J}\right)^{1/2} V^{3/4} \tag{2}$$

Since the ion current density J enters in this formula, we must estimate it. If the Debye sheath is to have a monotonic potential variation, the ion current entering it must exceed the value given by the Bohm criterion[1]

$$J \geq \frac{1}{2} n_o e \; (KTe/M)^{1/2} \, , \tag{3}$$

where M is the mass of a singly charged ion. This criterion will be rederived below. It implies that the ions are accelerated to an energy $\frac{1}{2}$ KTe by the electric field of a presheath that extends from deep inside the plasma to some ill-defined sheath edge, and that the density at the sheath edge is approximately $n_o/2$. The total sheath thickness is then

$$d = d_1 + d_2 \tag{4}$$

As a numerical example, take $n_o = 10^{12}$ cm^{-3}, KTe = 3eV, M = 40 M_H, and V = 100 volts. (V is a positive number; the potential -V is negative.) Then J = 21 mA/cm^2, $\lambda_D = 1.28 \times 10^{-3}$ cm, and $d_2 = 2 \times 10^{-2}$ cm. Thus $d = d_1 + d_2 = 2.3 \times 10^{-2}$ cm, and the Child-Langmuir sheath is 8 times thicker than the Debye sheath.

We now wish to calculate the combined sheath more exactly to check on the accuracy of this rough estimate. To do so requires an ad hoc assumption about the sheath edge, since strictly speaking the disturbance caused by a plane probe (or any probe in a strong magnetic field) extends to distances at which the probe no longer appears planar.

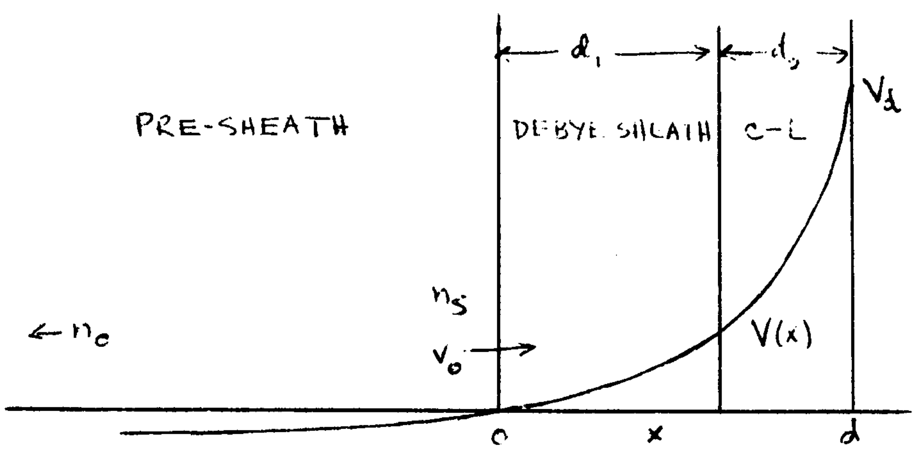


Figure 1.

Figure 1 shows the situation. The point designated as the sheath edge is labeled $x = 0$, and the potential V, for convenience, is defined to be 0 there. The sheath edge is defined to be the nearest point to the boundary at $x = d$ at which the plasma can still be considered to be quasineutral. Thus

$$n_e\,(x=0) = n_i\,(x=0) \equiv n_s. \tag{5}$$

If n_o is the density in the body of the plasma, it is intuitively clear that $n_s < n_o$. Since the electrons are Maxwellian (they are repelled by the sheath potential), the potential at $x<o$ must be positive relative to that at the sheath edge. This gives rise to the presheath electric field. Ions are accelerated toward the sheath by this field and cross the sheath edge with a drift velocity v_o. We shall neglect the thermal spread in ion velocities and assume that at $x = o$ there is a monoenergetic ion stream with energy

$$U = \frac{1}{2} M v_o^{\,2} \tag{6}$$

These assumptions will allow us to solve for V(x) and the sheath thickness. The arbitrary division into d_1 and d_2, shown in Figure 1, will not be necessary. The Debye sheath (d_1) is defined as the region where $n_i \neq n_e$ and n_e is not negligible. The Child-Langmuir sheath (d_2) is defined as the region where n_e is negligibly small. The difficult part is now finished; the rest of the problem is relatively straightforward.

For a one-dimensional system, the governing equations are:

$$\varepsilon_o \frac{d^2V}{dx^2} = n_i - n_e \qquad \text{(Poisson's equation)} \tag{7}$$

$$n_e = n_s e^{-eV/KT} \qquad \text{(Boltzmann relation)} \tag{8}$$

$$\frac{1}{2} M v_i^2 - eV = \frac{1}{2} M v_o^2 \qquad \text{(Energy conservation)} \tag{9}$$

$$n_i v_i = n_s v_o \qquad \text{(Ion continuity)} \tag{10}$$

Here it must be remembered that V is the negative of the potential and is positive. Eqs. (9) and (10) give

$$n_i = n_s \ (1 + 2eV/Mv_o^2)^{-\frac{1}{2}} \quad . \tag{11}$$

Eq. (7) then becomes

$$\varepsilon_o \frac{d^2V}{dx^2} = n_s \ [(1 + \frac{2eV}{Mv_o^2})^{-\frac{1}{2}} - e^{-eV/KT}]. \tag{12}$$

Near the sheath edge, V is small; and we may expand the right-hand side:

$$n_s^{-1} \ \varepsilon_o V'' \approx 1 - \frac{eV}{Mv_o^2} - 1 + \frac{eV}{KT} = eV \ (\frac{1}{KT} - \frac{1}{Mv_o^2}) \tag{13}$$

For small x>o, we must have V and V" both positive for the sheath to have a monotonic potential variation. Hence we require

$$U = \frac{1}{2} Mv_o^2 > \frac{1}{2} KT. \tag{14}$$

We may now define the sheath edge more specifically as the point where this condition is barely met; namely, where

$$v_o = (KT/M)^{1/2} \equiv v_B \tag{15}$$

This critical drift is referred to as the Bohm velocity v_B. To acquire this velocity, the ions must have fallen through a pre-sheath potential of magnitude $\frac{1}{2}KT/e$. Hence the density in the plasma, according to Eq. (8), is

$$n_o = n_s e^{-(e/KT)(-KT/2e)} = n_s e^{1/2} = 1.65 n_s, \tag{16}$$

or $n_s/n_o \approx 0.6$. This is the reason for the approximate factor $\frac{1}{2}$ in Eq. (3).

To solve the sheath equation, introduce the following dimensionless variables:

$$\eta \equiv eV/KT, \quad \xi = x/\lambda_{DS} \tag{17}$$

where λ_{DS} is the Debye length [Eq. (1)] evaluated with n_s instead of n_0.

Setting $v_0 = v_B$ in Eq. (12), we can write it as

$$\frac{d^2\eta}{d\xi^2} = (1 + 2\eta)^{-\frac{1}{2}} - e^{-\eta} \tag{18}$$

Multiplying by $\eta' = d\eta/d\xi$, we can integrate this once to give

$$\frac{1}{2}(\eta')^2 = (1 + 2\eta)^{\frac{1}{2}} + e^{-\eta} + C. \tag{19}$$

It is tempting to set the integration constant C equal to -2, so that $\eta' = 0$ at the sheath edge, where $\eta = 0$. However, this would restrict us to the trivial solution $\eta(\xi) = 0$ everywhere. The reason is that $\eta(0)$ and $\eta''(0)$ have been defined to be zero; and if $\eta'(0) = 0$, then $\eta'''(0)$ and all higher derivatives also vanish, as can be seen by differentiating Eq. (18). Thus we must allow $\eta'(0)$ to have a small but finite value s, determined by the pre-sheath scale length, as is clear from Figure 1. Eq. (19) then becomes

$$\eta' = 2^{1/2}\left[(1 + 2\eta)^{1/2} + e^{-\eta} - 2\right]^{1/2} + s. \tag{20}$$

The behavior of the two sheaths d_1 and d_2 (Fig. 1) can be recovered in the approximate limits. For $\eta \gg 1$, Eq. (20) becomes

$$\eta' \simeq 2^{1/2}(2\eta)^{1/4}. \tag{21}$$

This amounts to neglecting the electron density, the initial ion velocity, and the slope at the origin--exactly the conditions of the Child-Langmuir problem. Solving, we obtain

$$\eta^{-1/4} \eta' = \frac{4}{3} (\eta^{3/4})' = 2^{3/4}.$$

Integration from x = o to x = d yields

$$\eta^{3/4} = \frac{3}{4} 2^{3/4} \xi_d. \tag{22}$$

Aside from notation, this is precisely Eq. (2), the Child-Langmuir law.

In the limit of small η, we may expand the r.h.s. of Eq. (20) in Taylor series. The first non-vanishing term is third-order:

$$\eta' \simeq \frac{\sqrt{2}}{3} \eta^3 + s. \tag{23}$$

Thus

$$\xi = \int \frac{d\eta}{s + b\eta^3} \tag{24}$$

Using a table of integrals, we obtain an analytic form for the Debye sheath:

$$\xi = \frac{\alpha}{3s} \left[\frac{1}{2} \ln \frac{(\eta + \alpha)^2}{\eta^2 - \alpha\eta + \alpha^2} + \sqrt{3} \tan^{-1} \frac{2\eta - \alpha}{\alpha\sqrt{3}} \right], \tag{25}$$

where

$$\alpha \equiv (s/b)^{1/3}$$

$$b = \sqrt{2}/3$$

A smooth joining of the solutions (25) and (22) would give a reasonable approximation to the exact solution for any given value of the initial slope s. However, it is easier to integrate Eq. (20) numerically to study how the sheath thickness depends on s. We have done this on an HP-29C (or HP-25) hand calculator using the recursion formulas

$$\eta_n = \eta'_{n-1} \;\Delta\xi, \quad \eta'_{n-1} = f\,(\eta_{n-1}) \tag{26}$$

where $f(\eta)$ is the r.h.s. of Eq. (20).

The range of s was chosen by the following process. Let L be the scale length of the pre-sheath. If we consider the plane collector to be perpendicular to a dc magnetic field, then L would be the smallest of the ion mean free path, the ionization length, or the ion cross-field diffusion length. If the plane collector is parallel to the lines of force, it would be pulling ions across the magnetic field. Then L would be scaled to the ion Larmor radius evaluated with the electron temperature, since the pre-sheath potentials are of order KT_e/e. Taking the latter case, we have

$$L = \frac{M}{eB} \left(\frac{2KT_e}{M}\right)^{1/2}$$

and

$$s \propto \frac{\lambda_{DS}}{L} = \left(\frac{\varepsilon_o KT_e}{n_s e^2} \cdot \frac{M}{2KT_e}\right)^{1/2} \cdot \frac{eB}{M} \tag{27}$$

$$= (1.2)^{-1/2} \left(\frac{\varepsilon_o B^2}{n_o M}\right)^{1/2} = 0.91 V_A/C,$$

where V_A is the Alfvén speed, and we have used Eq. (16). For $B \simeq 2T$, $n_o \simeq 10^{18}\ m^{-3}$, and $M = 40\ M_H$, we find $s \simeq 2 \times 10^{-2}$.

Numerical results for the potential profile in the combined sheath are shown in Fig. 2 for s = .001, .01, and 0.1. It is seen that the joining of the small-η and large-η solutions could have been made smoothly around $\eta = 0.5$. On a semi-log plot, the curve is more S-shaped the smaller the value of s. This is because the Child-Langmuir sheath has a fixed scale length, but the Bohm or Debye sheath can have a very long scale length if s is chosen very small. In practice, the effective sheath thickness is not sensitive to the assumed value of s. At the distance where eV has fallen to a small fraction of KT, say 0.1KT, one is to all intents and purposes inside the plasma. If we define δ to be the distance between the $\eta = 0.1$ and $\eta = 100$ points, Figure 2 yields the following sheath thicknesses:

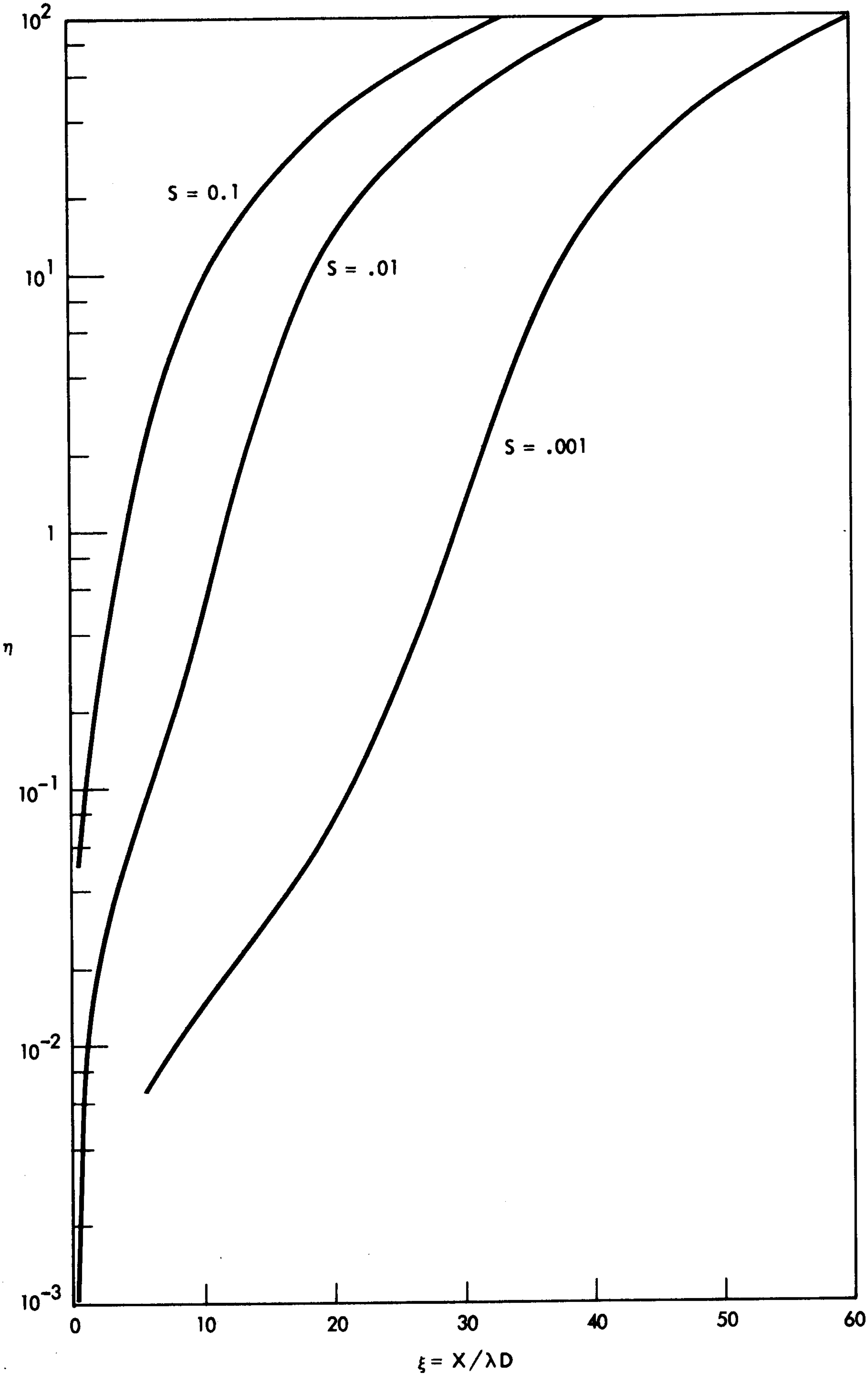
s	δ
.001	39 λ_{DS}
.01	36 λ_{DS}
0.1	33 λ_{DS}

Thus, the difference among the curves is mainly due to the choice of where in the pre-sheath one arbitrarily assumes the sheath to begin. The insensitivity of δ to s justifies the seemingly arbitrary assumptions on which Bohm-like calculations like this are based.

It is instructive to compare the calculated result with the rough estimate made at the beginning of this paper. For $KT_e = 3$ eV, $n_s = 0.6 \times 10^{12}$ and $\eta_d = 33$, we obtain $\delta = 3.3 \times 10^{-2}$ cm at s = .01. This is considerably larger than the 2.3×10^{-2} cm estimated earlier. The difference is probably due to the thickness of the transition region.

If the curves of Figure 2 are plotted on log-log paper, then one finds that the $d \propto V^{3/4}$ law is obeyed at both large η and small η, but the curves take a wiggle in between.

1. See F. F. Chen, in Plasma Diagnostic Techniques, ed. by R. S. Huddlestone and S. L. Leonard (Acad. Press, 1965) Chap. 4.



10^2
10^1
1
10^{-1}
10^{-2}
10^{-3}
η
S = 0.1
S = .01
S = .001
0
10
20
30
40
50
60
$\xi = X/\lambda D$