Task II-3224

PSP THEORY FOR EXPERIMENTALISTS, PART III

PLASMA RESPONSE TO INDUCED ELECTRIC FIELDS

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1. INTRODUCTION

This is the third paper in a series reviewing the plasma physics principles involved in the Dawson isotope separation process. In Part I (Task II-1359), we derived the simple fluid dispersion relation for the two-ion hybrid wave and showed how it changed continuously into the electrostatic ion cyclotron wave as $k_{||}$ was increased and electrons were allowed to move along B_0 to cancel space charge. The frequency shifts due to electron flow were calculated, and the main feature of the two-ion hybrid resonance was explained; namely, that the ac current of minor ions is independent of their concentration--a result of space charge neutralization requirements.

In Part II, entitled "Axial Eigenmodes for Long - $\lambda_{||}$ Waves in a Plasma Bounded by Sheaths" (Task II-2185), we considered the problem of electrostatic excitation of such waves by voltages applied to endplates. The shapes of resonant modes, corresponding to waves satisfying the dispersion relation, were found. The off-resonant response was also calculated. It was shown that even emissive endplates could not supply enough electron current through the sheath to help greatly in cancelling space charge. However, by exciting an antisymmetric mode with two endplates driven 180° out of phase, one can achieve neutralization of space charge by currents flowing entirely within the plasma.

In Part III (this paper), we consider the problem of inductive drive, in which an electric field is applied throughout the volume of the plasma. This problem has received considerable theoretical attention during the past three years. At first, the case of $k_{||} = 0$ (uniform excitation in the direction of $\underline{B}_0$), representing the most unfavorable space charge conditions, was treated. Field maxima at hybrid resonances and minima at cyclotron resonances were found; and the heights and widths of these resonances, which depend on thermal effects, were calculated analytically[1] and computationally[2]. The suppression of the left-hand circularly polarized component of $\underline{E}$ (the component that cyclotron-resonates with ions) at the major and minor species resonances has been verified experimentally[3]. More recently, the more practical case of finite $k_{||}$ has been investigated in cylindrical geometry by fluid calculations[4] and in plane geometry with particle simulations[5]. It was found that the application of a parallel component of $\underline{E}$ (in addition to finite $k_{||}$) not only allowed the field to penetrate easily into the plasma, an effect previously noticed in rf plugging of magnetic mirrors[6], but actually caused an enhancement of the applied field. The physical reason for this has been pointed out by J. M. Dawson. Our main purpose here is to reduce the problem to the simplest non-trivial case so as to obtain an analytic formula for the enhancement factor, and thus to exhibit its functional dependences.

The problem of space charge neutralization in a homogeneous plasma is, however, simple compared to that in a bounded or inhomogeneous plasma. The theory of drift waves should, in principle, automatically account for space charge in an inhomogenous plasma. We plan, in Part IV of this series, to apply drift wave theory to multi-species plasmas and show to what extent space charge effects are already incorporated in the proper radial eigenfunctions.

2. THE DIELECTRIC RESPONSE OF A PLASMA

2.1 FUNDAMENTAL EQUATIONS

Let an oscillating electric field $\underline{E}_0(\underline{r})$ be applied to a homogeneous plasma by currents in unspecified windings. The motion of charged particles in response to $\underline{E}_0$ will create a response field $\underline{E}_p$, so that the total field inside the plasma will be

$$\underline{E} = \underline{E}_0 + \underline{E}_p \tag{1}$$

The total field $\underline{E}$ obeys Maxwell's equations; written in e.s.u. for fields varying as exp $(-i\omega t)$, these are:

$$\underline{\nabla} \times \underline{E} = i\omega \underline{B} \tag{2}$$

$$c^2 \underline{\nabla} \times \underline{B} = 4\pi \, \underline{J} - i\omega \underline{E} \equiv - i\omega \underline{\underline{\varepsilon}} \cdot \underline{E} \tag{3}$$

The plasma current $\underline{J}$ is found from the equations of motion for the various species, and its proportionality to $\underline{E}$ can be expressed in terms of a conductivity tensor $\underline{\underline{\sigma}}$:

$$\underline{J} = \underline{\underline{\sigma}} \cdot \underline{E} \quad . \tag{4}$$

The dielectric tensor $\underline{\underline{\varepsilon}}$ (ω) [or $\underline{\underline{\varepsilon}}$ $(\omega, \underline{k})$ if KT $\neq$ 0] is therefore

$$\underline{\underline{\varepsilon}} = \underline{\underline{I}} + \frac{4\pi i}{\omega} \underline{\underline{\sigma}} \quad . \tag{5}$$

Substituting Eq. (3) into the curl of Eq. (2), we obtain

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \cdot \underline{E} \quad . \tag{6}$$

The vacuum field $\underline{E}_0$ (the applied field) is described by the same equation with $\underline{\underline{\varepsilon}}$ set equal to unity:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}_0) = \frac{\omega^2}{c^2} \underline{E}_0 \tag{7}$$

Subtracting Eq. (7) from Eq. (6) and using Eq. (1), we obtain

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}_p) = \frac{\omega^2}{c^2} \left(\underline{\underline{\varepsilon}} \cdot \underline{E}_p + \underline{\underline{\varepsilon}} \cdot \underline{E}_o - \underline{E}_o\right) \tag{8}$$

The response field $\underline{E}_p$ driven by $\underline{E}_o$ is therefore described by the equation

$$\nabla^2 \underline{E}_p - \underline{\nabla} (\underline{\nabla} \cdot \underline{E}_p) + \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}} \cdot \underline{E}_p = \frac{\omega^2}{c^2} (\underline{\underline{I}} - \underline{\underline{\varepsilon}}) \cdot \underline{E}_o . \tag{9}$$

2.2 ISOTROPIC PLASMA ($\underline{B}_o = 0$)

In an isotropic plasma, ε is a scalar; and one can define a scalar dielectric <u>function</u> K by

$$\underline{E} = E_o/K \tag{10}$$

As with ordinary dielectric materials, if $K > 1$, then the internal field $\underline{E}$ is lower than the applied field $\underline{E}_o$.

2.2.1 Electrostatic Waves

If $\underline{\nabla} \times \underline{E}_p = 0$, Eq. (8) for scalar ε yields

$$\underline{E}_p = \frac{1-\varepsilon}{\varepsilon} \underline{E}_o \tag{11}$$

so that

$$\underline{E} = \underline{E}_o + \underline{E}_p = \underline{E}_o/\varepsilon \tag{12}$$

In that case, K is the same as ε. When $\varepsilon = 0$, $\underline{E}_o \to 0$ for finite $\underline{E}$, which means that no external drive is needed to sustain an oscillation. Thus, $\varepsilon = 0$ is the dispersion relation for an undamped wave. Indeed, since $\varepsilon = 1 - \bar{\omega}_p{}^2/\omega^2$ when $B_o = 0$, we obtain the Langmuir wave dispersion relation $\omega^2 = \bar{\omega}_p{}^2$, where $\bar{\omega}_p{}^2 \equiv \omega_p{}^2 + \Omega_p{}^2$.

2.2.2 Electromagnetic Waves

If $\underline{\nabla} \cdot \underline{E}_p = 0$, Eq. (9) for scalar ε becomes

$$- \underline{\nabla} (\underline{\nabla} \cdot \underline{E}_p) + \frac{\omega^2}{c^2} \varepsilon \underline{E}_p = \frac{\omega^2}{c^2} (1-\varepsilon) \underline{E}_o \tag{13}$$

Assuming a uniform plasma, we may Fourier analyze in space, replacing $\underline{\nabla}$ by $i\underline{k}$:

$$-k^2 \underline{E}_p + \frac{\omega^2}{c^2} \varepsilon \underline{E}_p = \frac{\omega^2}{c^2} (1-\varepsilon) \underline{E}_o \tag{14}$$

Defining the index of refraction

$$\underline{\mu} \equiv c\underline{k}/\omega, \tag{15}$$

we obtain

$$\underline{E}_p = \frac{1-\varepsilon}{\varepsilon-\mu^2} \underline{E}_o \tag{16}$$

so that

$$\underline{E} = \underline{E}_o + \underline{E}_p = \frac{1-\mu^2}{\varepsilon-\mu^2} \underline{E}_o \tag{17}$$

Thus, for electromagnetic waves

$$K = \frac{\varepsilon-\mu^2}{1-\mu^2} \cdot \tag{18}$$

If the dispersion relation is satisfied, $\underline{E}_o \to 0$ and $K \to 0$ for finite $\underline{E}$; therefore, the dispersion relation is $\mu^2 = \varepsilon$ or $\omega^2 = \overline{\omega}_p^2 + c^2k^2$, as we expect for an electromagnetic wave for $B_o = 0$. Note that $K \to \infty$ for $\mu^2 \to 1$, an option that did not exist for electrostatic waves. This means that a vacuum wave $\mu^2 = 1$ will be completely shielded out, so that $\underline{E}_p = -\underline{E}_o$ and $\underline{E} = 0$. Suppose we now let $\varepsilon \to 0$. Then Eq. (18) indicates a finite K; yet we know that no electromagnetic waves can propagate if $\omega = \overline{\omega}_p$. The case $\varepsilon = 0$ must be treated separately; this is a consequence of the cold-plasma assumption.

2.2.3 The Case $\varepsilon = 0$.

Taking the divergence of Eq. (3), we obtain Poisson's equation

$$\underline{\nabla} \cdot (\underline{\underline{\varepsilon}} \cdot E) = 0. \tag{19}$$

for scalar ε, this becomes

$$\varepsilon (\underline{\nabla} \cdot \underline{E}) = 0, \tag{20}$$

so that either $\underline{\nabla} \cdot \underline{E} = 0$ or $\varepsilon = 0$. If $\underline{\nabla} \cdot \underline{E} = 0$, $\underline{\nabla} \cdot \underline{E}_p = 0$ also, since

$\underline{E}_o$ is a vacuum field. Hence, we recover the electromagnetic results of the previous section. If $\varepsilon = 0$ but $\underline{\nabla} \cdot \underline{E} \neq 0$, there are two cases. If $\underline{\nabla} \times \underline{E}_p = 0$ (electrostatic waves), we recover from Eq. (8) the electrostatic result of Eq. (12). On the other hand, if $\underline{\nabla} \times \underline{E}_p \neq 0$ (mixed e.s. and e.m. waves), Eq. (8) yields for $\varepsilon = 0$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}_p) = \frac{\omega^2}{c^2} \underline{E}_o. \tag{21}$$

Adding to Eq. (7), we obtain

$$\underline{E}_o + \underline{E}_p \equiv \underline{E} = 0. \tag{22}$$

Thus, waves with $\omega = \overline{\omega}_p$ are completely shielded out; and $K = 0$ in the limit $\varepsilon \to 0$, as one would expect.

2.3 ANISOTROPIC PLASMA ($B_0 \neq 0$)

In the case of a magnetically confined plasma, the plasma response is given by Eq. (9) with the dielectric tensor taking one of several well-known forms. We confine our attention to cold plasmas, in which $\underline{\underline{\varepsilon}}$ has, in the notation of Stix[7], the form

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}. \tag{23}$$

Here the elements S, D, and P are defined by the following equations:

$$S = \frac{1}{2}(R + L), \qquad D = \frac{1}{2}(R - L) \tag{24}$$

$$R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \pm \omega_{cs})} \tag{25}$$

$$L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega \mp \omega_{cs})} \tag{26}$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \tag{27}$$

where the sum extends over an arbitrary number of charged particle species $\underline{s}$, ω_{ps} and ω_{cs} are the plasma and cyclotron frequencies for species $\underline{s}$, and the upper (lower) sign is to be taken for positive (negative) species.

The nature of the plasma response can be illustrated by taking the simple example of an infinite, homogeneous plasma in Cartesian coordinates with $\underline{B}_o = B_o \hat{z}$ and

$$\underline{E}_o = E_o e^{i(kx-\omega t)} \hat{y}. \tag{28}$$

Thus $\underline{E}_o$ has only a y component and $\underline{k}$ only an x component, and the zz component of $\underline{\underline{\varepsilon}}$ is not required. In response to $\underline{E}_o$, particles will oscillate in the $\hat{y}$ direction; this causes no space charge because the field is uniform in this direction. However, because of the Lorentz force, particles tend to gyrate in cyclotron orbits and hence develop a motion in the $\hat{x}$ direction. Since $\underline{E}_o$ varies in x, space charge is built up, and an electrostatic field E_{px} is generated in response to the purely electromagnetic drive ($\underline{\nabla} \cdot \underline{E}_o = 0$). The total field (E_x, E_y) therefore has right- and left-hand circularly polarized components (E^R, E^L) even though $\underline{E}_o$ is plane polarized:

$$E^R = (E_x - iE_y)/\sqrt{2}, \qquad E^L = (E_x + iE_y)/\sqrt{2}\ . \tag{29}$$

We now solve for $\underline{E}_p$ from Eq. (9). From Eq. (23) we have

$$\underline{\underline{\varepsilon}} \cdot \underline{E}_p = \begin{pmatrix} S & -iD \\ iD & S \end{pmatrix} \begin{pmatrix} E_{px} \\ E_{py} \end{pmatrix} = \begin{pmatrix} SE_{px} & - & iDE_{py} \\ iDE_{px} & + & SE_{py} \end{pmatrix} \tag{30}$$

Using this in the x and y components of Eq. (9), and defining $\underline{\mu}$ as in Eq. (15), we obtain

$$-\mu^2 E_{px} + \mu^2 E_{px} + SE_{px} - iDE_{py} = iDE_o$$

$$-\mu^2 E_{py} + iDE_{px} + SE_{py} = (1-S)\, E_o \quad . \tag{31}$$

Solving for $\underline{E}_p$, we see that the electrostatic part of the plasma response is

$$E_{px} = iDE_o \; \frac{\mu^2-1}{S(\mu^2-S) + D^2} \;; \tag{32}$$

and the electromagnetic part is

$$E_{py} = - E_o \; \frac{S(1-S) + D^2}{S(\mu^2-S) + D^2} \tag{33}$$

The total field is found by adding E_o to the last equation, and the circularly polarized components are then obtained from Eq. (29). Using the identities

$$S^2 - D^2 = RL, \quad S + D = R, \quad S - D = L, \tag{34}$$

we finally obtain

$$E^R = - \frac{iE_o}{\sqrt{2}} \quad \frac{L\,(\mu^2-1)}{S\mu^2-RL} \tag{35}$$

$$E^L = \frac{iE_o}{\sqrt{2}} \quad \frac{R(\mu^2-1)}{S\mu^2-RL} \tag{36}$$

Both E^R and E^L become infinite when $S\mu^2-RL = 0$; indeed, the condition $\mu^2 = RL/S$ is just the dispersion relation for the extraordinary wave propagating across a magnetic field.[7] Conversely, $E^R = 0$ at the left-hand cutoff frequency[8] ω_L, where $L = 0$; and $E^L = 0$ at the right-hand cutoff frequency ω_R, where $R = 0$.

These frequencies are all in the electronic range. Of more interest to the present application are low frequencies in the ionic range, where $\mu^2 >> 1$. Since S is as large as R or L, we may neglect the term RL in the denominator, obtaining

$$E^R \simeq - L/S, \quad E^L \simeq R/S \tag{37}$$

At cyclotron resonance for an ion species, $L \to \infty$, according to Eq. (26); thus, $E^L \to 0$ at $\omega = \Omega_{cs}$, while E^R remains finite. This is because E^L rotates in the same direction as the ion gyration, and therefore the plasma response is so strong that the internal field is entirely shielded out. On the other hand, both E^R and E^L become large when S = 0, which is the condition for hybrid resonance[7]. With a single ion species, S = 0 yields the upper and lower hybrid frequencies ω_h and ω_ℓ[8], while with two ion species one has in addition the two-ion hybrid resonance frequency Ω_{ii}. In the latter case, one would expect E^L to vanish at the ion cyclotron frequencies Ω_1 and Ω_2 and to become large at the two-ion hybrid frequency in between. If the minor species 2 is much less abundant than the major species 1, one would expect the dip at Ω_2 to be much narrower than at Ω_1. Thus the left-hand component of $\underline{E}$ has the qualitative shape shown in Fig. 1.

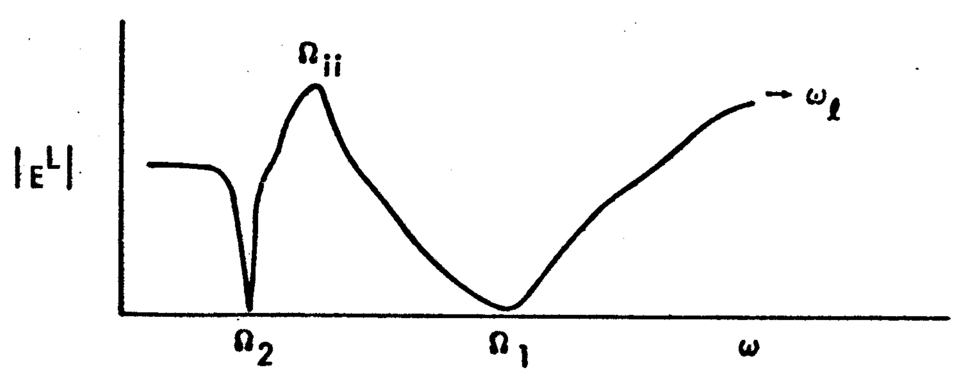

Fig. 1

Because of thermal motions and finite plasma effects, the zeroes and infinities in Fig. 1 will be smeared out, though their locations will not be changed. The computation of these effects in a finite length plasma with ion thermal motions is the essence of Ref. 1. The dips at Ω_{cs} do not reach zero because each ion sees a different Doppler-shifted frequency, and the peak at Ω_{ii} does not reach ∞ because no ion remains in the system long enough to feel a sharp resonance.

The above calculation can easily be generalized to an $\underline{E}_o$ field with both E_{ox} and E_{oy} components, as long as $E_{oz} = 0$. In this case the internal field is

$$E^L = \frac{E_{ox}}{\sqrt{2}} \frac{\mu^2 - R}{S\mu^2 - RL} + \frac{iE_{oy}}{\sqrt{2}} \frac{R\,(\mu^2-1)}{S\mu^2 - RL} \qquad (38)$$

$$E^R = \frac{E_{ox}}{\sqrt{2}} \frac{\mu^2 - L}{S\mu^2 - RL} - \frac{iE_{oy}}{\sqrt{2}} \frac{L\,(\mu^2-1)}{S\mu^2 - RL}. \qquad (39)$$

The E_{ox} component represents capacitive, electrostatic drive, but it does not alter the general behavior of the solution.

3. EXCITATION WITH CIRCULAR COILS

The preceding case is unrealistic in that both E_z and k_z are zero. We next allow k_z to be finite while keeping $E_z = 0$. We treat the Cartesian equivalent of inductive drive with purely circular coils, as shown in Fig. 2.

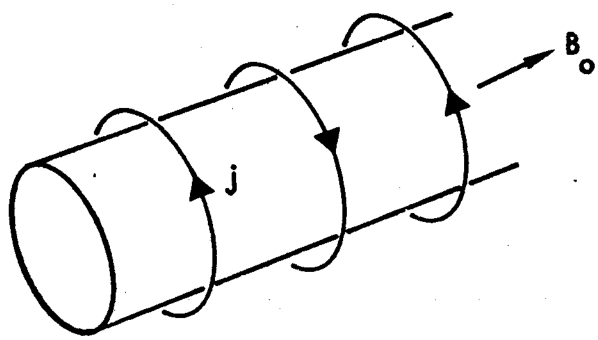

Fig. 2

The vacuum field has only a θ (or y) component but varies in both r (or x) and z. Since k has no θ component by symmetry, the condition $\underline{\nabla} \cdot \underline{E}_o = 0$ is automatically satisfied. Taking the lowest Fourier mode for simplicity, we may write

$$\underline{E}_o = E_o\, e^{i(k_x x + k_z z - \omega t)}\, \hat{y}. \tag{40}$$

Substituting this into our fundamental equation (9), we obtain from the x, y, and z components three simultaneous equations for E_{px}, E_{py}, and E_{pz}:

$$\begin{pmatrix} S-\mu_z^2 & -iD & \mu_x\mu_z \\ iD & S-\mu^2 & 0 \\ \mu_x\mu_z & 0 & P-\mu_x^2 \end{pmatrix} \begin{pmatrix} E_{px} \\ E_{py} \\ E_{pz} \end{pmatrix} = \begin{pmatrix} iD \\ 1-S \\ 0 \end{pmatrix} E_o, \tag{41}$$

where $\mu^2 \equiv \mu_x^2 + \mu_z^2$.

The determinant of the matrix is, from the middle line,

$$\mathbb{D} = (S-\mu^2)\,(SP - P\mu_z{}^2 - S\mu_x{}^2) - D^2(P - \mu_x{}^2) \tag{42}$$

Thus

$$\mathbb{D}E_{px} = \begin{Vmatrix} iD & -iD & \mu_x\mu_z \\ 1-S & S-\mu^2 & 0 \\ 0 & 0 & P-\mu_x^2 \end{Vmatrix} E_o,$$

and similarly for E_{py} and E_{pz}. The total field $\underline{E}$ is found by adding E_o to E_{py}, the other components being unaffected. We obtain

$$\begin{aligned} \mathbb{D}E_x &= E_o\; iD\; (1-\mu^2)\; (P-\mu_x^2) \\ \mathbb{D}E_y &= E_o(1-\mu^2)[(P-\mu_x^2)(S-\mu_z^2) - \mu_x^2\,\mu_z^2] \\ \mathbb{D}E_z &= E_o\; (-iD)\; (1-\mu^2)\; \mu_x\mu_z \end{aligned} \tag{43}$$

The circularly polarized components found from Eq. (29) are then

$$E^L = \frac{iE_o}{\sqrt{2}}\;\frac{R(\mu^2-1)\,(P-\mu_x^2) - P\mu_z^2\,(\mu^2-1)}{(S\mu^2-RL)\,(P-\mu_x^2) - P\mu_z^2\,(\mu^2-S)} \tag{44}$$

$$E^R = -\frac{iE_o}{\sqrt{2}}\;\frac{L(\mu^2-1)\,(P-\mu_x^2) - P\mu_z{}^2(\mu^2-1)}{(S\mu^2 - RL)\,(P-\mu_x^2) - P\mu_z^2\,(\mu^2-S)} \tag{45}$$

$$E_z = -iE_o\;\frac{D(\mu^2-1)\,\mu_x\mu_z}{(S\mu^2-RL)\,(P-\mu_x^2) - P\mu_z^2\,(\mu^2-S)} \tag{46}$$

where

$$\mu^2 \equiv \mu_x^2 + \mu_z^2 \tag{47}$$

This solution can easily be generalized to finite temperature plasmas. In the fluid approximation, one merely has to replace P by[9]

$$P = 1 - \sum_s \frac{\omega_{ps}{}^2}{\omega^2 - k_z^2 \; v_{ths}^2} \;, \tag{48}$$

where $$v_{ths}^2 = \gamma_s \, KT_s/M_s. \tag{49}$$

To include kinetic effects in the z-direction (e.g. Landau damping), one can replace P by

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{k_z^2 \; v_{ths}^2} \; Z'\left(\frac{\omega}{k_z v_{ths}}\right) \;, \tag{50}$$

where Z' is the derivative of the plasma dispersion function[9].

Of course, finite Larmor radius and other temperature effects perpendicular to $\underline{B}_o$ are not included in this simple approach.

When $\mu_z = 0$, Eqs. (44)-(46) can be seen to reduce to Eqs. (35)-(36), as expected. However, for $\omega \lesssim \Omega_p$, P is extremely large, as can be seen from Eq. (48). In fact, since $P \gtrsim M/m$, the $P\mu_z^2$ terms in Eqs. (44)-(46) will dominate for $\mu_z > (m/M)^{1/2}$ unless one of the other quantities is resonant. This is just to say that electron charge neutralization by motion along $\underline{B}_o$ is so efficient that it completely changes the nature of the wave motions.

Consider the dispersion relation, found by setting the denominator $\mathcal{D}$ equal to zero. This yields

$$\tan^2 \theta = - \frac{P(\mu^2-R)\;(\mu^2-L)}{(S\mu^2-RL)\;(\mu^2-P)} \;, \tag{51}$$

where $\tan \theta = \mu_x/\mu_z$. Eq. (51) can be recognized[7] as the general dispersion relation for a cold plasma. If we let $\mu^2 \to \infty$ to specialize to slow electrostatic waves, we obtain

$$\tan^2 \theta = - P/S, \tag{52}$$

which is just the resonance angle formula, which leads to the Gould-Trivelpiece modes in finite geometry. If we further take $\theta = 0$ and use the warm-plasma expression Eq. (48) for P, the condition P=0 yields the Bohm-Gross and ion acoustic waves propagating along $\underline{B}_o$. At intermediate angles, electrostatic ion cyclotron waves can be obtained. Thus, peaks in E^L and E^R can occur not only at Ω_{ii} and ω_ℓ, but also at many other frequencies satisfying the dispersion relation, depending on the angle θ.

We now consider the vicinity of the ion cyclotron resonances, where $L \to \infty$. In this limit, Eqs. (44) and (46) show that E^L and $E_z \to 0$, since the denominator becomes large with L, and hence S, while the numerator remains finite. E^R, however, contains L in the numerator as well as in the denominator and hence approaches a finite value. As a result, the component with the wrong polarization can penetrate into the plasma, but the component that can provide ion cyclotron acceleration is cancelled by the strong ion response. The introduction of k_z through the use of circular excitation coils has not changed the fact that $E^L = 0$ at the ion cyclotron frequencies.

Note, however, that the ion fluid velocity does not vanish with E^L, for it must remain finite in order to generate the response field $E_p^{L'}$ that cancels the applied field. For species s, the left circularly polarized velocity component v_s^L is easily found from the fluid equation of motion to be

$$V_s^L = \frac{ie}{M_s} \frac{E^L}{\omega - \Omega_{cs}} \quad . \tag{53}$$

From Eqs. (44) and (26), we see that E^L near $\omega = \Omega_{cs}$ is proportional to

$$E^L \propto \frac{1}{L} \approx \left[\frac{\Omega_{ps}^2}{\omega(\omega - \Omega_{cs})}\right]^{-1} . \tag{54}$$

Hence

$$V_s^L \propto n_{os}^{-1} , \tag{55}$$

so that the cyclotron current $n_{os}q_sV^L_s$ of the species s remains finite when $E^L \to 0$, and furthermore does not depend on how large the density n_{os} is. However, the width of the resonance decreases with n_{os}, and the effective acceleration of a minor species in the presence of transit time and Doppler broadening effects requires finding a mechanism for introducing a finite E^L component into the plasma.

4. EXCITATION WITH HELICAL COILS -- PHYSICAL MECHANISM

To force an E^L component into the plasma, one has to induce a component E_{oz}, as can be done with a helical coil or with circular coils connected by conductor segments aligned with the B-field. The basic effect is illustrated in Fig. 3.

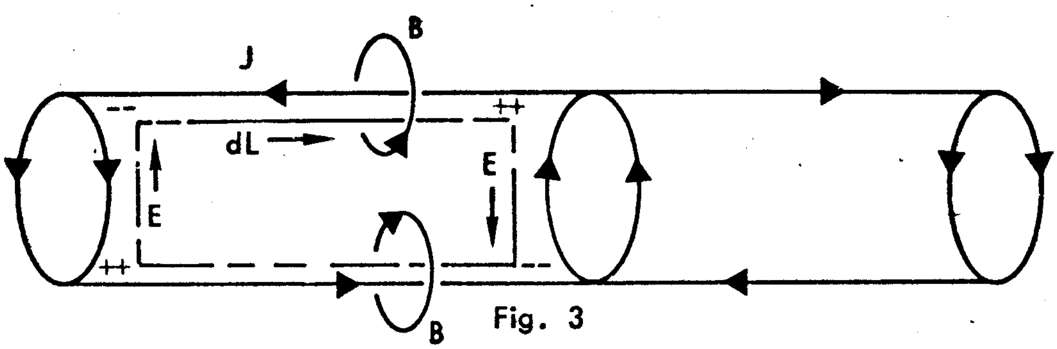

Fig. 3

Here the excitation is by an rf current in a periodic array of split circular coils connected by rods running in the z direction. As the current J increases, a magnetic field B is created as shown, inducing an electric field in the plasma. Faraday's Law applied to the dotted path says

$$\oint \underline{E} \cdot d\underline{L} = - \int \dot{\underline{B}} \cdot d\underline{S} \qquad (56)$$

Since the coils are periodic in z, the induced field will have finite k_z. In this case, the good mobility of electrons along $\underline{B}$ will cause E_z to vanish. Hence, the contribution to the line integral comes entirely from the vertical legs of the path, and there must exist vertical electric fields as shown. These are created by electrostatic charge build-ups caused by electrons moving along B. The applied E_{oz} component causes electrons to move, to the left on the top leg and to the right on the bottom leg, until their space charge causes an electrostatic E_{pz} opposite to E_{oz} and just sufficient to cancel it. The space charge, however, also creates an electrostatic E_y which is <u>in the same direction</u> as the E_{oy} induced by the current in the circular coils. Hence a plane-polarized E_y component exists in the body of the plasma, and part of this is an E^L circularly polarized component that is useful. Of course, the right-hand side of Eq. (56) is reduced by the B-field due to currents flowing in the plasma (in the opposite direction to the excitation current J); but the cancellation cannot be exact under all plasma conditions, because the current required to establish a given electrostatic field depends on the plasma dielectric constant.

A more realistic coil is the bifilar (m=2) helical structure shown in Fig. 4.

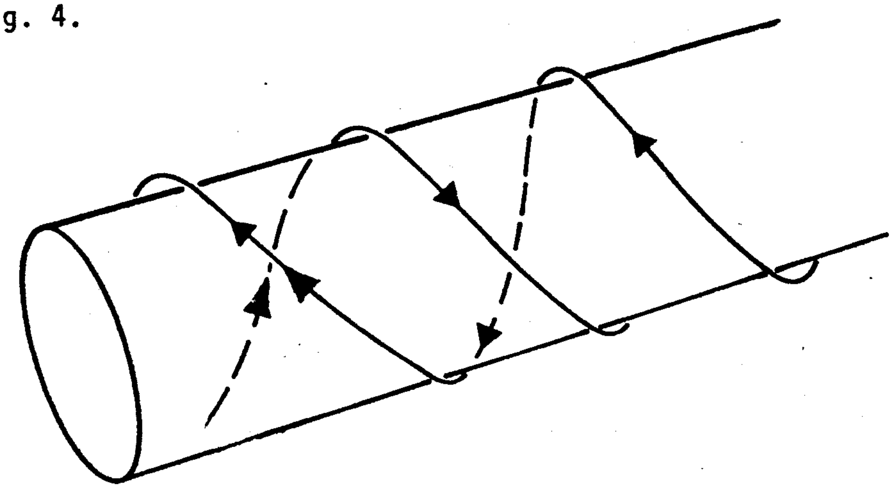

Fig. 4

We wish to simplify the problem by treating an equivalent slab geometry. Note that all three components of $\underline{k}$ are needed: k_z is needed to cause the periodic electron charge pile-ups; k_θ (or k_y) is needed to close the current loop in the excitation structure; and k_r (or k_x) exists because the fields must vary radially. The vacuum field $\underline{E}_o$ contains all three components in cylindrical geometry but is simpler in plane geometry, where $E_{ox} = 0$. The plasma response E_{px} is, of course, essential and must be retained.

The simplest plane-geometry problem containing all the essential elements is shown in Fig. 5.

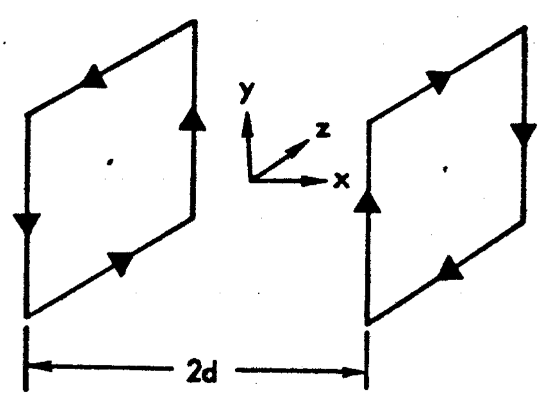

Fig. 5

The external current flows in an infinite periodic array of rectangular coils on the planes $x = \pm d$. The current density varies sinusoidally in both the y and z directions, with wavenumbers k_y and k_z, respectively. The currents on opposite faces are opposite, and the origin is halfway between nulls in these current distributions. Since $\underline{E} = \underline{\nabla}\phi - \dot{\underline{A}}$ and $\underline{A}$ is parallel to $\underline{J}$, the induced vacuum field ($\phi = 0$) has no x component as long as $\underline{J}$ has no x component.

The configuration of Fig. 5 is not quite equivalent to the helical coil of Fig. 4. On opposite sides of the plasma in Fig. 5, the exciting currents are equal and opposite. In Fig. 4, however, opposite points (for instance, those where heavy arrows are drawn) have J_z in the opposite directions but J_y in the same direction (though J_θ is indeed opposite). This causes a 90° phase shift between induced and electrostatic components of $\underline{E}$ in the two problems, as will be seen explicitly; but no large difference in the magnitude of E will result.

Watari et al. treat a simpler looking geometry (Fig. 6):

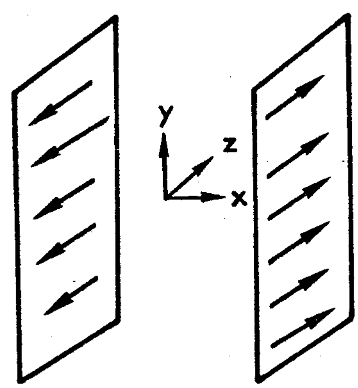

Fig. 6

Here the excitation current is in the z direction only, and the system is infinite and homogeneous in the y direction. To allow electrostatic charges to build up, the plasma is given a finite length in the z direction. This geometry is included in our treatment as the case $k_y \to 0$, but the formulation is not as neat because the abrupt end of the plasma causes all k_z harmonics to be generated, and the current system requires sources located at infinity.

5. PLASMA RESPONSE TO INDUCED FIELDS WITH FINITE E_z

Before attempting the problem of a plasma slab or cylinder, we calculate the response of an infinite plasma to an induced E-field periodic in all three Cartesian directions. Let the driving field be

$$\underline{E}_o = (E_o \hat{y} + E_{oz} \hat{z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}, \tag{57}$$

where $\hat{z}$ is the direction of $\underline{B}_o$. Since this is an induced field, $\underline{\nabla} \cdot \underline{E}_o = 0$; hence,

$$E_{oz} = -\,(k_y/k_z)\, E_o. \tag{58}$$

Because of the electrostatic charges that build up in response to E_{oz}, the field generated by the plasma has all three components:

$$\underline{E}_p = (E_x\hat{x} + E_y\hat{y} + E_z\hat{z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}. \tag{59}$$

The field $\underline{E}_o$ is not one that can be produced in practice, for it requires a non-interacting current distribution in the y-z plane inside the plasma:

$$\underline{j}_o = (j_{oy}\hat{y} + j_{oz}\hat{z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}. \tag{60}$$

We must show that this current has no effect on our basic equation, Eq. (9).

For the vacuum field, Maxwell's equations give

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E}_o = i\omega\, \underline{\nabla} \times \underline{B}_o = 4\pi i\omega c^{-2}\, \underline{j}_o + k_0^2\, \underline{E}_o, \tag{61}$$

where $k_o \equiv \omega/c$. The total field $\underline{E}$ in the plasma is given by

$$\underline{\nabla} \times \underline{\nabla} \times \underline{E} = i\omega\, \underline{\nabla} \times \underline{B} = i\omega c^{-2} \left(4\pi \underline{j}_o + \underline{\underline{\varepsilon}} \cdot \dot{\underline{E}}\right), \tag{62}$$

where the plasma currents are included in $\underline{\underline{\varepsilon}}$, and the excitation current $\underline{j}_o$ is written explicitly. With $\underline{E} = \underline{E}_o + \underline{E}_p$, Eq. (62) becomes

$$\underline{\nabla} \times \underline{\nabla} \times (\underline{E}_o + \underline{E}_p) = 4\pi i\omega c^{-2}\, \underline{j}_o + k_o^2 \left(\underline{\underline{\varepsilon}} \cdot \underline{E}_o + \underline{\underline{\varepsilon}} \cdot \underline{E}_p\right). \tag{63}$$

Subtracting Eq. (61) from this yields

$$- \underline{\nabla} \times \underline{\nabla} \times \underline{E}_p + k_o^2\, \underline{\underline{\varepsilon}} \cdot \underline{E}_p = k_o^2 \left(\underline{\underline{I}} - \underline{\underline{\varepsilon}}\right) \cdot \underline{E}_o, \tag{64}$$

which is the same as Eq. (9).

We wish to solve Eq. (9) assuming an $\underline{E}_p$ of the form (59) under the assumption of large electron mobility along $\underline{B}_o$:

$$E_z^{tot} = E_{oz} + E_z \simeq 0, \quad E_z \simeq -E_{oz} = (k_y/k_z)\, E_o \tag{65}$$

However, if we straightforwardly substitute Eq. (59) into Eq. (9) and write out the x and y components of Eq. (9), using Eq. (65) for E_z and Eq. (23) for $\underline{\underline{\varepsilon}}$, we fall into a trap. The two simultaneous equations for E_x and E_y are found to be as follows, with $\underline{\mu} \equiv \underline{k}/k_o$:

$$\begin{pmatrix} S-1+\mu_x^2 & \mu_x\mu_y - iD \\ \mu_x\mu_y + iD & S-1+\mu_y^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = - \begin{pmatrix} \mu_x\mu_y - iD \\ S-1+\mu_y^2 \end{pmatrix} E_o \tag{66}$$

Since the second column on the left is the negative of the column multiplying E_o, the determinant solution of Eq. (66) obviously yields

$$E_x = 0, \quad E_y = -E_{oy} \;\therefore\; \underline{E}^{tot} = 0. \tag{67}$$

There appears to be perfect shielding of the applied field. This trivial solution can be avoided by removing the assumption (65) and allowing $\varepsilon_{zz} = P$ to be finite. However, the real trouble lies in the disappearance of the electrostatic component of $\underline{E}_p$: Since the $\underline{E}_p$ we obtained is $-\underline{E}_o$ and $\underline{E}_o$ is divergence free, $\underline{E}_p$ is also divergence free. If we compare with the circular-coil result of Eqs. (44)-(46), we see that in the limit $P \to \infty$ that solution gives $E_z/E_o = 0$, $E^R/E_o^R = E^L/E_o^L \simeq 1$ (assuming $\mu_z^2 >> S$, R, L). However, that is reasonable for the circular-coil case, where no field amplification due to electrostatic charges can occur.

To retain the electrostatic component, we divide $\underline{E}_p$ into transverse and longitudinal parts:

$$\underline{E}_p = \underline{E}^t + \underline{E}^\ell, \text{ where } \underline{\nabla} \cdot \underline{E}^t = 0 \text{ and } \underline{\nabla} \times \underline{E}^\ell = 0. \tag{68}$$

Eq. (64) now becomes

$$\underline{\mu} \times (\underline{\mu} \times \underline{E}^t) + \underline{\underline{\varepsilon}} \cdot (\underline{E}^t + \underline{E}^\ell) = (\underline{\underline{I}} - \underline{\underline{\varepsilon}}) \cdot \underline{E}_o$$

or

$$-\mu^2 \underline{E}^t + \underline{\underline{\varepsilon}} \cdot (\underline{E}^t - \underline{\nabla}\phi) = (\underline{\underline{I}} - \underline{\underline{\varepsilon}}) \cdot \underline{E}_o, \tag{69}$$

where we have introduced the scalar potential ϕ:

$$\underline{E}^\ell = - \underline{\nabla}\phi. \tag{70}$$

$\underline{E}_z^t$ can be eliminated from Eq. (69) by the condition $\underline{\nabla} \cdot \underline{E}^t = 0$:

$$E_z^t = -\mu_z^{-1} (\mu_x E_x^t + \mu_y E_y^t). \tag{71}$$

Also, the condition $\underline{E}_z^t + \underline{E}_z^\ell + \underline{E}_{oz} = 0$ allows us to eliminate ϕ:

$$-i\mu_z \phi = (\mu_y/\mu_z) E_o + \mu_z^{-1} (\mu_x E_x^t + \mu_y E_y^t). \tag{72}$$

The x and y components of Eq. (60) then can be written as two simultaneous equations in E_x^t and E_y^t. Using the notation of Eq. (23) for the components of $\underline{\underline{\varepsilon}}$, we obtain

$$\begin{bmatrix} (S-\mu^2)\mu_z^2 + \mu_x(S\mu_x - iD\mu_y) & -iD\mu_z^2 + \mu_y(S\mu_x - iD\mu_y) \\ iD\mu_z^2 + \mu_x(S\mu_y + iD\mu_x) & (S-\mu^2)\mu_z^2 + \mu_y(S\mu_y + iD\mu_x) \end{bmatrix} \begin{bmatrix} E_x^t \\ E_y^t \end{bmatrix}$$

$$= -E_0 \begin{bmatrix} -iD\mu_z^2 + \mu_y(S\mu_x - iD\mu_y) \\ (S-1)\mu_z^2 + \mu_y(S\mu_y + iD\mu_x) \end{bmatrix} \qquad (73)$$

Here

$$\mu^2 = \mu_x^2 + \mu_y^2 + \mu_z^2 >> 1 \qquad (74)$$

Note that the r.h.s. is the same as the second column of the matrix on the left except that $(S-\mu^2)$ is replaced by $(S-1)$. The trivial solution $\underline{E}_p = -\underline{E}_0$ is avoided only because the internal driving current $\underline{j}_0$ forces a short wavelength in the plasma, such that $\mu^2 >> 1$, or $\underline{k} >> \underline{k}_0$. This point is important in understanding the next section.

To solve Eq. (73), we write the lower term on the r.h.s. as

$$(S-\mu^2)\,\mu_z^2 + \mu_y(S\mu_y + iD\mu_x) + (\mu^2 - 1)\,\mu_z^2 .$$

Now Eq. (73) takes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E_x^t \\ E_y \end{pmatrix} = -E_0 \begin{pmatrix} b \\ d+\delta \end{pmatrix} \qquad (75)$$

where

$$\delta \equiv (\mu^2 - 1)\,\mu_z^2, \qquad (76)$$

and a, b, c, d are defined by looking at Eq. (73). The determinant

$$\mathbf{D}_* \equiv \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \qquad (77)$$

can be simplified to the following form with the identity $S^2 - D^2 \equiv RL$:

$$\mathbf{D}_* = -\mu^2\mu_z^2\,[S\mu^2 - RL + \mu_z^2(S - \mu^2)] \equiv -\mu^2\mu_z^2\,\mathbf{D} . \qquad (78)$$

The solution of Eq. (73) can then be written:

$$\mathbb{D}_* E_x^{\ t} = -E_0 \begin{Vmatrix} b & b \\ d+\delta & d \end{Vmatrix} = b\delta E_o \tag{79}$$

$$\mathbb{D}_* E_y^{\ t} = -E_0 \begin{Vmatrix} a & b \\ c & d+\delta \end{Vmatrix} = -(\mathbb{D}_* + a\delta)\, E_o . \tag{80}$$

This yields

$$\begin{aligned} \mathbb{D}E_x^{\ t} &= E_o[(1-\mu^{-2})\ [-S\mu_x\mu_y + iD\ (\mu^2 - \mu_x^{\ 2})] \\ \mathbb{D}E_y^{\ t} &= E_o[(1-\mu^{-2})\ (S\mu_x^{\ 2} - iD\mu_x\mu_y\ ) - (S\mu^2 - RL) - (S-\mu^2)\ (\mu_z^{\ 2}/\mu^2)]\ . \end{aligned} \tag{81}$$

The electrostatic potential ϕ is given by Eq. (72), and the gradient of this gives the longitudinal field $\underline{E}^{\ell}$. After some cancellations, we obtain

$$\begin{aligned} \mathbb{D}E_x^{\ \ell} &= E_o(1-\mu^{-2})\mu_x[(S-\mu^2)\mu_y + iD\mu_x] \\ \mathbb{D}E_y^{\ \ell} &= E_o(1-\mu^{-2})\mu_y\ [(S-\mu^2)\ \mu_y + iD\mu_x] \\ \mathbb{D}E_z^{\ \ell} &= E_o(1-\mu^{-2})\mu_z[(S-\mu^2)\ \mu_y + iD\mu_x] = -\mathbb{D}(E_{oz} + E_z^{\ t}). \end{aligned} \tag{82}$$

When $\underline{E}^t$, $\underline{E}^{\ell}$, and $\underline{E}_o$ are summed, further cancellations result in these simple expressions for the total field in the plasma:

$$\begin{aligned} E_x^{\ tot} &= \mathbb{D}^{-1}\ E_o\ (\mu^2 - 1)\ (iD - \mu_x\mu_y) \\ E_y^{\ tot} &= \mathbb{D}^{-1}\ E_o\ (\mu^2 - 1)\ (S - \mu_y^{\ 2} - \mu_z^{\ 2}) \\ E_z^{\ tot} &= 0 \qquad \text{(by assumption)} \end{aligned} \tag{83}$$

Where $$\mathbb{D} = S\mu^2 - RL + \mu_z^{\ 2}\ (S-\mu^2) \tag{84}$$

The circularly polarized components are

$$E_o^{L,R} = \pm i\, E_o / \sqrt{2}, \qquad E^{L,R} = (E_x \pm i\, E_y)/\sqrt{2} \tag{85}$$

The total internal field components can then be written:

$$\frac{E^R}{E_o^R} = (\mu^2-1)\, \frac{L - \mu_y^2 - \mu_z^2 - i\mu_x\mu_y}{S\mu^2 - RL + \mu_z^2\,(S-\mu^2)} \tag{86}$$

$$\frac{E^L}{E_o^L} = (\mu^2-1)\, \frac{R - \mu_y^2 - \mu_z^2 + i\mu_x\mu_y}{S\mu^2 - RL + \mu_z^2\,(S-\mu^2)} \tag{87}$$

Eqs. (86) and (87) show the amplification of the applied field that is possible, and Eq. (82) gives explicitly the electrostatic field that is responsible for this effect.

For example, consider an argon plasma of density 10^{12} cm^{-3} with B_o = 20 kG and $\omega \simeq \Omega_c$. At these low frequencies, S, R, and L are all about equal to $1 + c^2/V_A^2 \simeq 2 \times 10^3$, and $k_o \simeq 1.6 \times 10^{-4}$ cm^{-1}. Let $k_x \simeq k_y \simeq 2\pi/10$ cm^{-1} and $k_z \simeq k_y/5$. Then $\mu^2 \simeq 5 \times 10^7$, so that the inequalities $\mu^2 \gg S \gg 1$ are well satisfied. Except at cyclotron resonance, Eqs. (86) and (87) are approximately

$$\frac{E^{R,L}}{E_o^{R,L}} \simeq \frac{-\mu_y^2 - \mu_z^2 \mp i\mu_x\mu_y}{-\mu_z^2} \simeq \sqrt{2}\, \frac{\mu_y^2}{\mu_z^2}\ . \tag{88}$$

The amplification factor is seen to be purely geometrical and amounts to $\simeq 35$ in this example, where $k_x \simeq k_y \simeq 5k_z$. The plasma dielectric has no effect because β is too low. Only when β is high enough that the plasma currents alter the flux-conservation argument of Fig. 3 will the off-resonance amplification factor be affected. In the circular-coil case of $E_{oz} = 0$, amplification cannot occur. This can be seen by taking the $P \rightarrow \infty$ limit of Eqs. (44) and (45), which become

$$\frac{E^{R,L}}{E_0^{R,L}} \simeq \frac{-P\mu_z^2\mu^2}{-P\mu_z^2\mu^2} = 1 \tag{89}$$

The imaginary term in Eq. (88) can be traced back to Eq. (83), where it is clear that E_x and E_y are nearly in phase (because the space charge pulls simultaneously in all directions), and therefore the field is close to linear polarization.

The field enhancement effect has been called plasma paramagnetism by the group[6] in Nagoya, Japan, who have observed this effect in rf plugging experiments. However, in a recent paper[10] the Japanese seem to have missed the point about the electrostatic nature of the field; they calculate the internal magnetic field and use magnetic probes to measure it, thus ensuring that the main enhancement will be missed.

In the region of ion cyclotron resonance, L goes to infinity and S to L/2. Eqs. (87) and (86) then show that E^L goes to zero while E^R remains finite. Screening of the field by the plasma at exact resonance cannot be avoided, but the minimum will have finite width and depth because of thermal Doppler shifts and transit time effects.

6. INDUCED FIELDS IN A PLASMA SLAB

We finally come to the more realistic problem of fields induced in a plasma by currents in external conductors. To keep the algebra as simple as possible, we consider a single Fourier mode in slab geometry, as shown in Fig. 7. The plasma is uniform and infinite in the y and z directions,

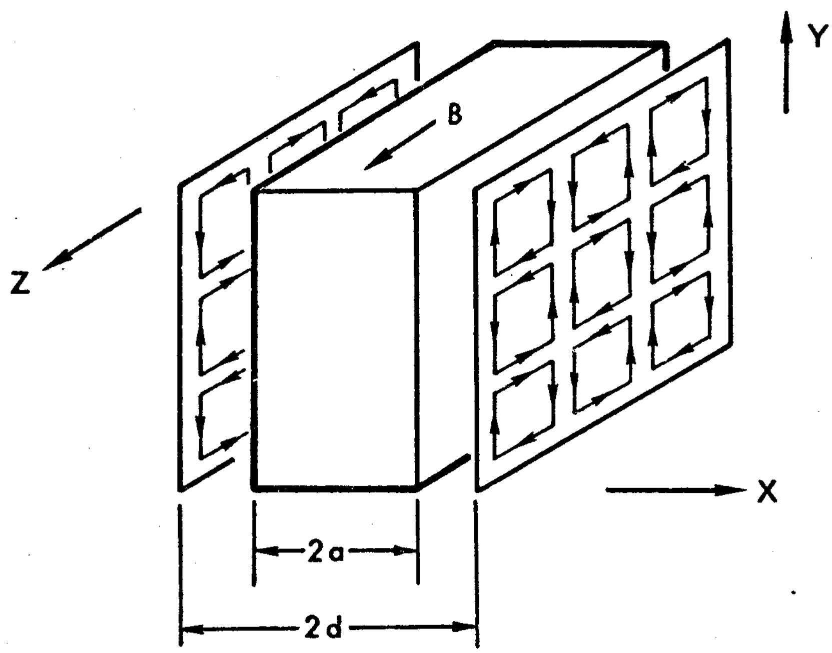

Figure 7

and has a thickness 2a in the x direction. The exciting currents flow on two sheet conductors at $x = \pm d$. The currents vary sinusoidally in the y and z directions and are either in phase on the opposite sheet (symmetric excitation) or out of phase (antisymmetric excitation). With the origin at the center of the diagram, the surface current $\underline{K}_0\,(y,z)$ for the antisymmetric case can be written

$$\underline{K}_0 = \pm 4K_0 e^{-i\omega t} \left(\frac{\hat{y}}{\mu_y} \cos k_y y \sin k_z z - \frac{\hat{z}}{\mu_z} \sin k_y y \cos k_z z \right), \text{ at } x = \pm d \qquad (90)$$

It will be convenient to use exponential notation. If we allow k_y and k_z to have positive and negative values, Eq. (90) can be expressed as the sum of four waves of the type Re [exp i ($\pm$ $k_y y$ $\pm$ $k_z z$-ωt)] with different combinations of sign. Treating one of these at a time, we can write

$$\underline{K}_o = \pm K_o \left(\frac{\hat{y}}{\mu_y} - \frac{\hat{z}}{\mu_z} \right) e^{i(k_y y + k_z z)} e^{-i\omega t} \quad \text{at } x = \pm d \qquad (91)$$

The coefficients have been chosen to satisfy $\underline{\nabla} \cdot \underline{K}_o = 0$.

The vacuum field $\underline{E}_o$ due to these currents satisfies

$$\nabla^2 \underline{E}_o + k_o^2 \underline{E}_o = 0. \qquad (92)$$

Assuming $\underline{E}_o \propto \exp\ i\ (k_y y + k_z z)$, we find solutions proportional to exp ($\pm$ $k_x x$), where

$$k_x^2 = k_y^2 + k_z^2 - k_o^2 . \qquad (93)$$

The combination corresponding to symmetric excitation is obviously cosh $k_x x$, and to anti-symmetric excitation, sinh $k_x x$. Furthermore, $\underline{E}_o$ cannot have an x component because $K_{ox} = 0$ (this is not true for the r component in cylindrical geometry) and the y and z component must satisfy $\underline{\nabla} \cdot \underline{E} = o$. Hence $\underline{E}_o$ has the form

$$\underline{E}_o = E_o \left(\frac{\hat{y}}{\mu_y} - \frac{\hat{z}}{\mu_z} \right) e^{-i\omega t} e^{i(k_y y + k_z z)} \sinh k_x x \qquad (94)$$

for the anti-symmetric case, and similarly with cosh $k_x x$ for the symmetric case. To evaluate E_o in terms of K_o, we compute the induced magnetic field at the surface of the conductors. The result, which will be derived in the next section, is

$$E_o = \frac{4\pi i}{e\mu_x} K_o e^{-k_x d} \qquad \text{(anti-sym.)} \qquad (95)$$

and similarly, with sinh $k_x d$, for the symmetric case. For completeness, we have also considered the case of shifted coils, in which K_{oy} is in phase at $x = \pm d$ but K_{oz} is out of phase. This mocks up the situation with a double helical coil (Fig. 4). However, it cannot be done in plane geometry without current feeds all over, since $\underline{K}_o$ is no longer divergenceless. The vacuum field in this case is given by

$$\underline{E}_o = E_o \left(\frac{\hat{y}}{\mu_y} \cosh k_x x - \frac{\hat{z}}{\mu_z} \sinh k_x x - \frac{\hat{x}}{\mu_x} e^{-k_x x} \right) e^{-i\omega t} e^{i(k_y y + k_z z)} \tag{96}$$

To calculate the plasma response to $\underline{E}_o$, we must solve Eq. (9). As in the case of distributed excitation current, a straightforward solution with $E_z^{tot} = 0$ yields only the trivial solution $\underline{E}_p = -\underline{E}_o$, or $\underline{E}^{tot} = 0$. Hence we split $\underline{E}_p$ into transverse and longitudinal parts, as in Eqs. (68) - (70), and attempt to solve the equation

$$\nabla^2 E^t + \underline{\underline{\varepsilon}} \cdot (\underline{E}^t - \underline{\nabla}\phi) = (\underline{\underline{\mathbb{I}}} - \underline{\underline{\varepsilon}}) \cdot \underline{E}_o. \tag{97}$$

Though this seems to be a straighforward extension of the problem in Sec. 5, it is not. Since the x-dependence is not constrained to be periodic, the present problem is much more complicated; it involves both the penetration of $\underline{E}_o$ into the plasma and the response of the plasma to whatever local field exists.

Before launching into the solution of Eq. (97), we wish to discuss two apparently divergent points of view on this problem. The first is that the driving field E_0 must surely extend into the plasma almost as far as it does in vacuum, because E_0 is divergenceless, and it can be cancelled only by another divergenceless field generated by plasma currents. But in a low-β plasma the magnetic field prevents large transverse currents from flowing (except at cyclotron resonances), and space charge prevents large periodic parallel currents at low frequency. In this case, it should be possible to calculate the plasma response to a given $\underline{E}_0$ in the same way as was done in Sec. 5. The second point of view is that followed in standard calculations of rf excitation by coils[7,11]. Here a wave launched by the coil is considered to impinge on the surface of the plasma. Waves satisfying the plasma dispersion relation are excited at the plasma surface and either propagate inwards (if $\underline{k}$ is real) or evanesce inwards (if $\underline{k}$ is imaginary). If the plasma thickness is just right, a column resonance can be excited. The plasma currents and charges then generate fields which radiate back into the vacuum region (scattered field). By matching the proper components of $\underline{E}^{tot}$ and its derivative to the plasma field at the plasma boundary, where $\underline{E}^{tot}$ includes the driving field and the scattered field, and applying appropriate boundary conditions at the coil and the vacuum chamber wall, the fields everywhere can be obtained. This approach seems to neglect near-field phenomena, in which an internal $\underline{E}_0$ can drive a plasma response that does not satisfy the dispersion relation; yet the results[11] of this method of calculation seem to agree with the enhancement factor found in Eq. (88). We shall attempt to reconcile these two points of view.

First, let us try to solve Eq. (97). The x and y components are (for anti-symmetric excitation):

$$\begin{aligned} \nabla^2 E_x{}^t + S(E_x{}^t - \phi') - iD(\dot{E}_y{}^t - i\mu_y\phi) &= iDE_0\mu_y{}^{-1}\sinh k_x x \\ \nabla^2 E_y{}^t + iD(E_x{}^t - \phi') + S(E_y{}^t - i\mu_y\phi) &= (1 - S)E_0\mu_y{}^{-1}\sinh k_x x \ , \end{aligned} \tag{98}$$

where the prime indicates $k_0^{-1}\ \partial/\partial x$. We can eliminate ϕ by means of the conditions

$$\underline{\nabla} \cdot \underline{E}^t = E_x^{t'} + i\mu_y E_y{}^t + i\mu_z E_z{}^t = 0 \tag{99}$$

$$E_z{}^{tot} = E_{oz} + E_z{}^t - i\mu_z\phi = 0 \tag{100}$$

This results in two coupled differential equations for E_x and E_y (we now suppress the superscript t):

$$(\mu_z^2 - S)E_x'' - D\mu_y E_x' + \mu_z^2(S - \mu_y^2 - \mu_z^2)E_x - iS\mu_y E_y' - iD(\mu_y^2 + \mu_z^2)E_y$$

$$= iS\mu_x E_o \cosh k_x x + iD(\mu_y^2 + \mu_z^2)E_o \mu_y^{-1} \sinh k_x x \tag{101}$$

$$\mu_z^2 E_y'' + D\mu_y E_y' + (\mu_y^2 + \mu_z^2)(S - \mu_z^2)E_y - iDE_x'' - iS\mu_y E_x' + iD\mu_z^2 E_x$$

$$= -D\mu_x E_o \cosh k_x x + [\mu_z^2 - S(\mu_y^2 + {}_z^2)]\, E_o \mu_y^{-1} \sinh k_x x,$$

where, from Eq. (93),

$$\mu_x^2 = \mu_y^2 + \mu_z^2 - 1\,. \tag{102}$$

In Eq. (101), the inhomogeneous terms on the r.h.s. have x-dependences of the form sinh $k_x x$ and cosh $k_x x$; hence, if we expand E_x and E_y as a series of terms in sinh $nk_x x$ amd cosh $mk_x x$, only the terms n=m=1 have non-vanishing coefficients. Hence we take E_x and E_y to be of the form

$$E_x = a \cosh k_x x + b \sinh k_x x$$

$$E_y = c \cosh k_x x + d \sinh k_x x \tag{103}$$

Substituting into Eq. (101) and equating the coefficients of the cosh $k_x x$ and sinh $k_x x$ terms, we obtain four coupled equations for the coefficients a, b, c, and d:

$$\begin{bmatrix} \mu_z^2(S-1) - S\mu_x^2 & -D\mu_x\mu_y & -iD(\mu_y^2+\mu_z^2) & -iS\mu_x\mu_y \\ -D\mu_x\mu_y & \mu_z^2(S-1) - S\mu_x^2 & -iS\mu_x\mu_y & -iD(\mu_y^2+\mu_z^2) \\ -iD(\mu_x^2-\mu_z^2) & -iS\mu_x\mu_y & \mu_z^2(S-1) + S\mu_y^2 & D\mu_x\mu_y \\ -iS\mu_x\mu_y & -iD(\mu_x^2-\mu_z^2) & D\mu_x\mu_y & \mu_z^2(S-1) + S\mu_y^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} iS\mu_x\mu_y \\ iD(\mu_y^2 + \mu_z^2) \\ -D\mu_x\mu_y \\ -[\mu_z^2(S-1) + S\mu_y^2] \end{bmatrix} \frac{E_o}{\mu_y} \tag{104}$$

We see that the last column on the left is proportional to the r.h.s. Hence the determinant solution of Eq. (104) gives 0 for all coefficients except d, which is $-E_0/\mu_y$. Eq. (103) then says that $E_x{}^t = 0$ and $E_y{}^t = -(E_0/\mu_y) \sinh k_x x = -E_{oy}$. Substituting into Eqs. (99) and (100), we see that $\phi = 0$. Hence this is the spurious solution of Eq. (97) in which the electrostatic effect does not appear and the transverse field completely cancels the applied field.

Next we attempt to force a non-trivial solution of Eq. (9) by iterating around a solution which, on intuitive grounds, has to be approximately correct. This solution is the one obtained by neglecting the transverse plasma field $\underline{E}^t$ and keeping only the electrostatic response $\underline{E}^\ell$. Clearly, $\underline{E}^t$ must be small since it is generated by plasma currents, and these are severely limited in the perpendicular direction by the magnetic field (except at cyclotron resonance) and in the parallel direction by space charge buildup.

If we neglect $\underline{E}^t$, the condition for perfect conductivity along $\underline{B}_0$ can be written

$$E_z{}^\ell = -E_{oz} - E_z{}^t \approx -E_{oz} \qquad (105)$$

or from Eq. (94),

$$-ik_z\phi = \mu_z{}^{-1}E_o \sinh k_x x. \qquad (106)$$

Hence

$$\phi(x) = iE_o(\sinh k_x x)/k_o\mu_z{}^2 \qquad (107)$$

This potential gives rise to the following electrostatic field components:

$$\begin{aligned} E_x{}^\ell &= -\phi' = -i(\mu_x/\mu_z{}^2)E_o \cosh k_x x \\ E_y{}^\ell &= -ik_y\phi = (\mu_y/\mu_z{}^2)E_o \sinh k_x x \end{aligned} \qquad (108)$$

The total internal field in the y direction is, from Eq. (94),

$$E_y^{tot} = E_y^{\ell} + (E_o/\mu_y) \sinh k_x x = [1 + (\mu_y^2/\mu_z^2)] E_{oy}. \quad (109)$$

The enhancement factor is therefore approximately

$$1 + (k_y^2/k_z^2) , \quad (110)$$

independent of plasma parameters, in agreement with what we found in Sec. 5.

However, the components E_x, E_y in Eq. (108) do not satisfy Eq. (9). For $\underline{\nabla}\mathrm{x}\underline{E}_p = 0$, the latter can be written

$$\underline{\underline{\varepsilon}} \cdot \underline{E}_p = (\underline{\underline{\mathrm{I}}} - \underline{\underline{\varepsilon}}) \cdot \underline{E}_o , \quad (111)$$

or

$$\begin{aligned} SE_x - iDE_y &= iD(E_o/\mu_y) \sinh k_x x \\ SE_y + iDE_x &= (1 - S)(E_o/\mu_y) \sinh k_x x \end{aligned} \quad (112)$$

Clearly Eq.(108) does not satisfy this, and a correction in the form of a transverse field $\underline{E}^t$ must be added. We propose the following iteration procedure. The approximate solution, Eq.(107) or (108), is inserted for $\underline{\nabla}\phi$ in Eq.(97), and that equation is solved for E_x^t and E_y^t. From the condition $\underline{\nabla}\cdot\underline{E}^t = 0$, we can calculate E_z^t and use that in Eq. (105) to get an improved value for $\phi(x)$. The process can be repeated.

We have carried out this procedure with disappointing results. After the first iteration, one indeed obtains an "improved" value for ϕ; namely,

$$\phi(x) = \frac{i E_o \sinh k_x x}{k_o \mu_z^2} \left\{ 1 - \frac{1}{\mu_z^2} \left[1 + \frac{(S-1)(\mu_z^2 - 1)}{2S-RL-1} \right] \right\} \quad (113)$$

However, the process does not converge uniformly. The next iteration yields an expression for ϕ that leads to $E_y^{tot} = 0$; hence, we are led back to the spurious solution where the driving field is entirely shielded out.

The trick of separating $\underline{E}_p$ into longitudinal and transverse components, which worked so well in the case of distributed driving currents, has failed to give us a non-trivial solution when the driving currents are outside the plasma. To understand what is happening, we return to the fundamental equation, Eq.(9), which (in units of k_o^{-1}) can be written

$$\nabla^2 \underline{E}_p - \underline{\nabla} (\underline{\nabla} \cdot \underline{E}_p) + \underline{\underline{\varepsilon}} \cdot \underline{E}_p = (\mathrm{I\!I} - \underline{\underline{\varepsilon}}) \cdot \underline{E}_o \tag{114}$$

With the condition $E_{pz} = -E_{oz}$, the components of $\underline{\underline{\varepsilon}}$ given in Eq.(23), and the components of $\underline{E}_o$ given in Eq.(94), this yields the coupled differential equations

$$\begin{aligned} (S - \mu_y^2 - \mu_z^2) E_x - iDE_y - i\mu_y E_y' &= iD\mu_y^{-1} E_o \sinh k_x x + i\mu_x E_o \cosh k_x x \\ E_y'' + (S - \mu_z^2) E_y + iDE_x - i\mu_y E_x' &= \mu_y^{-1} E_o (1 - S - \mu_y^2) \sinh k_x x \,, \end{aligned} \tag{115}$$

where the prime stands for $k_o^{-1}\, \partial/\partial x$.

These are linear, inhomogeneous equations with the driving terms on the right. We can let $\underline{E}_p$ be the sum of a particular solution and a solution to the homogeneous equations. The only particular solution we have been able to find is $\underline{E}_p = -\underline{E}_o$; this is what we find when we solve Eq.(115) straight forwardly assuming that E_x and E_y are composed of $\sinh k_x x$ and $\cosh k_x x$ terms. When this solution is subtracted out from Eq.(115), we are left with an E-field that has to satisfythe homogeneous set

$$\begin{aligned} (S - \mu_y^2 - \mu_z^2) E_x - iDE_y - i\mu_y E_y' &= 0 \\ E_y'' + (S - \mu_z^2) E_y + iDE_x - i\mu_y E_x' &= 0 \end{aligned} \tag{116}$$

We can Fourier analyze E_x, E_y so that

$$E_x \,,\, E_y \propto \exp(i\gamma x), \tag{117}$$

where γ is an internal wave number not necessarily related to k_x.

Eq.(116) then becomes

$$
\begin{aligned}
(S - \mu_y^2 - \mu_z^2)\, E_x + (\gamma\mu_y - iD)\, E_y &= 0 \\
(\gamma\mu_y + iD)\, E_x + (S - \mu_z^2 - \gamma^2)\, E_y &= 0
\end{aligned}
\tag{118}
$$

A solution requires the determinant $\mathbb{D}$ of the coefficients to vanish. Thus γ must satisfy

$$
(S - \mu_y^2 - \mu_z^2)(S - \mu_z^2 - \gamma^2) = \gamma^2 \mu_y^2 + D^2 . \tag{119}
$$

This is, of course, just the dispersion relation for waves in the plasma with $E_z = 0$; and the condition $\mathbb{D} = 0$ is exactly the condition under which the straightforward determinant solution of Eq.(115) is invalid (because of division by 0). So we are naturally led to natural modes of oscillation in the plasma rather than locally driven modes. Of course, γ can turn out to be real or imaginary. If γ is real, there will be finite-plasma resonance when the thickness of the plasma slab matches the natural wavelength; if γ is imaginary, the modes will be evanescent at the driving frequency. (γ cannot be complex because we have for simplicity neglected damping terms.)

The dispersion relation (119) easily reduces to

$$
\gamma^2 = \frac{RL - S\mu_z^2}{S - \mu_z^2} - (\mu_y^2 + \mu_z^2) \tag{120}
$$

When $\mu^2 >> S, R, L$, as is usual except near cyclotron resonance, we see that $\gamma^2 \approx -(\mu_y^2 + \mu_z^2)$, and hence the waves are indeed evanescent. The e-folding length is set by the periodicity of the external coils. To be more exact, we can make the standard low-frequency approximation ($\omega^2 < \Omega_c^2 << \omega_c^2$), obtaining

$$
S \approx 1 - \frac{\Omega_p^2}{\omega^2 - \Omega_c^2}, \quad RL = \left(1 + \frac{\Omega_p^2/\Omega_c}{\Omega_c + \omega}\right)\left(1 + \frac{\Omega_p^2/\Omega_c}{\Omega_c - \omega}\right) \approx S^2. \tag{121}
$$

Eq.(120) then becomes

$$\gamma^2 \approx S - \mu_y^2 - \mu_z^2 . \tag{122}$$

For the case of the Nagoya Type III coil ($\mu_y = 0$), good penetration ($\gamma \approx 0$) requires

$$\frac{c^2 k_z^2}{\omega^2} = 1 - \frac{\Omega_p^2}{\omega^2 - \Omega_c^2} \tag{123}$$

Except for our neglect of KT_e, this is identical with the condition for good field enhancement given by Watari et al.[6] in their Eq.(13). In the limit of low density ($\Omega_p \to 0$, $S \to 1$), Eq.(122) becomes [c.f. Eq.(93)]

$$\gamma^2 = 1 - \mu_y^2 - \mu_z^2 = -\mu_x^2,$$

and we recover the vacuum-field decay length. Propagating waves ($\gamma^2 > 0$) require $S > \mu_y^2 + \mu_z^2$, or

$$\frac{\Omega_p^2}{\Omega_c^2 - \omega^2} > (k_y^2 + k_z^2)/k_0^2 , \tag{124}$$

where the "1" in Eq.(121) has been neglected. Thus ω must be close to Ω_c and just below it. When finite temperature is taken into account, ω can be above Ω_c; and, in fact, Eq.(122) simply becomes the dispersion relation for an electrostatic ion cyclotron wave.

Thus we have been forced to consider waves in the plasma which satisfy the dispersion relation. The amplitude of these waves cannot be found from the homogeneous equations, of course. If undamped cyclotron waves are excited, they can reach arbitrary amplitude. If evanescent waves are excited, their amplitudes can be found by matching the plasma solution to the vacuum solution at the plasma boundary. We have therefore shown that calculating the plasma response to a local induced electric field is entirely equivalent to calculating the surface excitation of natural plasma modes by a launched wave impinging on the plasma. The latter method is the only method feasible in the present case, which explains why it is the standard procedure.

7. SOLUTION BY STANDARD METHODS

To obtain a working formula for the field enhancement factor, we now solve the excitation problem using the standard method of finding solutions in the different regions and matching them at the boundaries. The procedure is tedious, but it cannot be avoided. Let a uniform plasma extend from $x = -a$ to $x = a$, and let a perfectly conducting boundary lie at $x = \pm b$, as shown in Fig. 8.

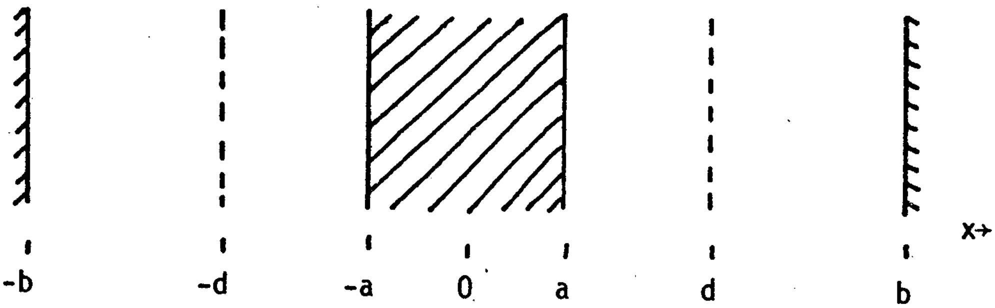

Figure 8

Antisymmetric sheets of noninteracting, periodic currents lie at $x = \pm d$, as in Figure 7. The sheet currents are as in Eq. (91):

$$\underline{K}_o = \pm K_o \left(\frac{\hat{y}}{\mu_y} - \frac{\hat{z}}{\mu_z} \right) e^{i(k_y y + k_z z - \omega t)} \tag{125}$$

At this point one has to be careful about the use of exponential notation for the y and z spatial variations; indiscriminate use of Eq. (125) leads to erroneous results. Suppose we wish to represent sinusoidal current distributions whose current maxima are lined up with the origin as shown in Fig. 9.

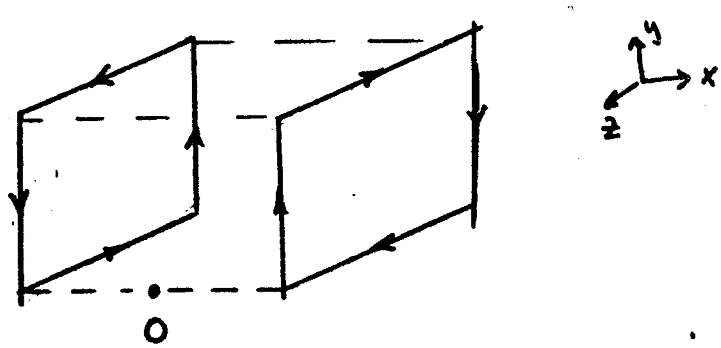

Fig. 9

The current sheets at $x = \pm d$ then have components

$$K_{oy} = \pm \frac{K_o'}{|\mu_y|} \sin |k_y| y \ \cos |k_z| z$$

$$K_{oz} = \mp \frac{K_o'}{|\mu_z|} \sin |k_z| z \ \cos |k_y| y \tag{125a}$$

Now let K_{oy} have the form of Eq. (125) and sum over the four possible combinations $k_y = \pm |k_y|$, $k_z = \pm |k_z|$. At $x = +d$, we have

$$K_{oy} = \frac{K_o}{|\mu_y|} e^{i(|k_y|y + |k_z|z)} - \frac{K_o}{|\mu_y|} e^{-i(|k_y|y - |k_z|z)}$$

$$+ \frac{K_o}{|\mu_y|} e^{i(|k_y|y - |k_z|z)} - \frac{K_o}{|\mu_y|} e^{-i(|k_y|y + |k_z|z)}$$

$$= \frac{4iK_o}{|\mu_y|} \sin|k_y|y \ \cos |k_z|z.$$

Thus the sum yields Eq. (125a) if we set $K_o' = 4iK_o$. A similar conclusion holds for K_{oz} and for $x = -d$. Consequently, we may use the exponential form (125) if, at the end, a sum is taken over the four waves with $\mu_y = \pm |\mu_y|$, $\mu_z = \pm |\mu_z|$.

In the exterior region $d < x < b$, $\underline{E}^e$ satisfies $\nabla^2 \underline{E}^e + \underline{E}^e = 0$, where lengths are normalized to k_o^{-1}. Hence $\underline{E}^e$ is of the form $\exp(\pm\mu_x x)$, where

$$\mu_x^2 = \mu_y^2 + \mu_z^2 - 1 \tag{126}$$

Since E_y^e and E_z^e must vanish at $x = b$ (for brevity we write the formulas only for positive x), they must be of the form

$$E_y^e = A_y^e e^{\mu_x x} \left[1 - e^{2\mu_x (b-x)} \right]$$

$$E_z^e = A_z^e e^{\mu_x x} \left[1 - e^{2\mu_x (b-x)} \right] . \tag{127}$$

The condition $\underline{\nabla} \cdot \underline{E}^e = 0$ gives

$$E_x^{\ e} = -i\mu_x^{-1}(\mu_y A_y^{\ e} + \mu_z A_z^{\ e}) e^{\mu_x x} \left[1 + e^{2\mu_x(b-x)}\right] \tag{128}$$

which need not vanish since it is normal to the conducting boundary. Thus $\underline{E}^e$ is expressed in terms of two undetermined coefficients, $A_y^{\ e}$ and $A_z^{\ e}$.

In the middle region $a < x < d$, $\underline{E}^m$ also satisfies $\nabla^2\underline{E}^m + \underline{E}^m = 0$ and so is of the form exp $(\pm\mu_x x)$, with μ_x satisfying Eq. (126). The component $E_z^{\ m}$ must vanish at $x = a$ because we assume $E_z = 0$ inside the plasma. Thus it will be convenient to use the following linear combination of exponentials:

$$\begin{aligned} E_x^{\ m} &= A \sinh \xi + B \cosh \xi \\ E_y^{\ m} &= C \sinh \xi + D \cosh \xi \\ E_z^{\ m} &= E \sinh \xi \quad , \end{aligned} \tag{129}$$

where

$$\xi \equiv \mu_x(x - a) \tag{130}$$

The condition $\underline{\nabla} \cdot \underline{E}^m = 0$ allows us to equate coefficients of sinh ξ and of cosh ξ to obtain

$$E = \mu_z^{-1}(i\mu_x B - \mu_y C) \tag{131}$$

$$D = i(\mu_x/\mu_y)A. \tag{132}$$

Matching the tangential components of $\underline{E}^e$ and $\underline{E}^m$ at $x = d$ yields

$$\begin{aligned} A_y^{\ 0} &= \frac{C \sinh \xi_d + A(i\mu_x/\mu_y) \cosh \xi_d}{e^{\mu_x d}\left[1 - e^{2\mu_x(b-d)}\right]} \\ A_z^{\ 0} &= \frac{(i\mu_x B - \mu_y C) \sinh \xi_d}{\mu_z e^{\mu_x d}\left[1 - e^{2\mu_x(b-d)}\right]} \end{aligned} \tag{133}$$

where

$$\xi_d \equiv \mu_x(d - a) \quad .$$

The undetermined coefficients at this point are A, B, and C. The normal component E_x must also be continuous at $x = d$ because there is neither a jump in ε nor a surface charge there. C does not appear in this matching condition, and one obtains

$$B = -A \frac{\sinh \xi_d + \cosh \xi_d \operatorname{ctnh} \zeta}{\cosh \xi_d + \sinh \xi_d \operatorname{ctnh} \zeta} \tag{134}$$

where

$$\zeta \equiv \mu_x(b - d) \ . \tag{135}$$

The $[B_n]$ jump condition yields nothing new, but $[B_{tan}]$ yields

$$\begin{aligned} [cB_y] &= \mu_z(E_x^{\,e} - E_x^{\,m}) + i(E_z^{\,e'} - E_z^{\,m'}) = (4\pi/c)K_{oz} \\ [cB_z] &= \mu_y(E_x^{\,e} - E_x^{\,m}) + i(E_y^{\,e'} - E_y^{\,m'}) = (4\pi/c)K_{oy} \ . \end{aligned} \tag{136}$$

The E_x terms cancel since E_x is continuous, and the two equations (136) yield the same result for C:

$$C = \left[\frac{4\pi i K_o}{c\mu_x\mu_y} - iA\,\frac{\mu_x}{\mu_y}\,(\sinh \xi_d + \cosh \xi_d \operatorname{ctnh} \zeta)\right](\cosh \xi_d + \sinh \xi_d \operatorname{ctnh} \zeta)^{-1} \tag{137}$$

Using Eqs. (131), (132), (134) and (137), we can write Eq. (129) in terms of the single coefficient A:

$$\begin{aligned} E_x^{\,m} &= A(\sinh \xi - F \cosh \xi) \\ E_y^{\,m} &= \frac{i\mu_x}{\mu_y}\left[(\mu_x^{-2}\,\overline{K}_o - FA) \sinh \xi + A \cosh \xi\right] \\ E_z^{\,m} &= -i(\mu_x\mu_z)^{-1}\,\overline{K}_o \sinh \xi \ , \end{aligned} \tag{138}$$

where

$$\overline{K}_o \equiv \frac{4\pi}{c}\frac{K_o}{M}, \quad F \equiv N/M \tag{139}$$

$$N \equiv \sinh \xi_d + \cosh \xi_d \operatorname{ctnh} \zeta \tag{140}$$

$$M \equiv \cosh \xi_d + \sinh \xi_d \operatorname{ctnh} \zeta \tag{141}$$

We now consider the interior field $\underline{E}^i$ in the plasma region. This must satisfy the homogeneous equation

$$\nabla^2 \underline{E}^i - \underline{\nabla}(\underline{\nabla} \cdot \underline{E}^i) + \underline{\underline{\varepsilon}} \cdot \underline{E}^i = 0 , \tag{142}$$

which we have previously evaluated in Eq. (116) for the case $E_z{}^i = 0$. The solution is of the form exp ($\pm gx$), where $g = i\gamma$ satisfies the dispersion relation (120):

$$g^2 = \mu_y{}^2 + \mu_z{}^2 + \frac{S\mu_z{}^2 - RL}{S - \mu_z{}^2} \tag{143}$$

In the case of antisymmetric excitation, it is clear that $E_y{}^i$ will be antisymmetric and thus vary as sinh gx. The electrostatic part $E_x{}^i$, however, will be symmetric because of the way the space charge accumulates (see Fig. 7). Thus we let

$$\begin{aligned} E_x{}^i &= A_x{}^i \cosh gx \\ E_y{}^i &= A_y{}^i \sinh gx \\ E_z{}^i &= 0 . \end{aligned} \tag{144}$$

Matching $E_y{}^i$ to $E_y{}^m$ yields

$$A_y{}^i \sinh ga = iA\mu_x/\mu_y \tag{145}$$

We cannot evaluate $[D_n]$ and match E_x because an unknown surface charge can develop in the surface of this ideal dielectric. However, no surface current can exist on a dielectric, so all components of $\underline{B}$ must be continuous at $x = a$. The $[B_x]$ condition yields nothing new. The $[B_y]$ condition yields

$$A_x^{\,i} \cosh ga = \mu_z^{-2} \overline{K}_o - AF \; . \tag{146}$$

After some manipulation involving Eq. (145), the $[B_z]$ condition yields

$$A = \frac{\overline{K}_o(1 + \mu_y^{\,2}/\mu_z^{\,2})}{\mu_x^{\,2}F + g\mu_x \operatorname{ctnh} ga} \; . \tag{147}$$

Putting all this together, we finally obtain

$$E_y^{\,i} = \frac{4\pi i}{c} K_{oy} \left(1 + \frac{\mu_y^{\,2}}{\mu_z^{\,2}}\right) \frac{\sinh gx}{N\mu_x \sinh ga + Mg \cosh ga} \tag{148}$$

$$E_x^{\,i} = \frac{4\pi}{c} \frac{K_{oz}}{\mu_z M} \left(\frac{1 - g\mu_x (M/N) \operatorname{ctnh} ga}{\mu_x^{\,2} + g\mu_x (M/N) \operatorname{ctnh} ga}\right) \frac{\cosh gx}{\cosh ga} \; , \tag{149}$$

where N and M are defined by Eqs. (140-141), g satisfies Eq. (143), and ξ_d and ζ are defined by Eqs. (133) and (135).

The boundary at $x = b$ is an unnecessary complication; if we let $b \to \infty$, we find $\zeta = \infty$, $\operatorname{ctnh} \zeta = 1$, and

$$N = M = \exp\left[\mu_x (d - a)\right] \; . \tag{150}$$

For $b \to \infty$ we have

$$E_y^{\,i} = \frac{4\pi i}{c} K_{oy} \left(1 + \frac{\mu_y^{\,2}}{\mu_z^{\,2}}\right) \frac{e^{-\mu_x (d-a)} \sinh gx}{\mu_x \sinh ga + g \cosh ga} \tag{151}$$

$$E_x^{\,i} = \frac{4\pi}{c} \frac{K_{oz}}{\mu_z} e^{-\mu_x (d-a)} \left(\frac{1 - g\mu_x \operatorname{ctnh} ga}{\mu_x^{\,2} + g\mu_x \operatorname{ctnh} ga}\right) \frac{\cosh gx}{\cosh ga} . \tag{152}$$

$$E_z^{\,i} = 0. \tag{153}$$

We note that this is finite when d = a, so we could have put the coils on the surface of the plasma. On the other hand, if b = d so that $\zeta = 0$, the field $\underline{E}^i$ would have to be zero, according to Eqs. (148) and (149). This is because the image currents on the conducting plate would have cancelled the applied currents.

To find the enhancement factor, we need to know the vacuum field $\underline{E}_o$. This cannot be found by taking the limit $\omega_p^2 \to 0$, S, R, L $\to$ 1 because the approximation $E_z^i = 0$ breaks down when there are no electrons to short out E_z. From Eqs. (143) and (126) we see that when S = R = L = 1, we have

$$g^2 = \mu_y^2 + \mu_z^2 - 1 = \mu_x^2.$$

This is as expected, but there is no way the finite value of E_{oz} or the zero value of E_{ox} can be obtained from a limit of Eqs. (152) and (153).

We now solve for the vacuum field $\underline{E}_o$ when $b \to \infty$. For x<d, symmetry requires $\underline{E}_o$ to be of the form

$$\begin{aligned} E_{ox}^i &= A_{ox}^i \cosh \mu_x x \, e^{i(\mu_y y + \mu_z z - \omega t)} \\ E_{oy}^i &= A_{oy}^i \sinh \mu_x x \, e^{i(\mu_y y + \mu_z z - \omega t)} \\ E_{oz}^i &= A_{oz}^i \sinh \mu_x x \, e^{i(\mu_y y + \mu_z z - \omega t)} \end{aligned} \qquad (154)$$

where μ_x satisfies Eq. (126). For x>d, only solutions finite at infinity are acceptable, so we can write

$$\begin{aligned} E_{ox}^e &= A_{ox}^e \, e^{-\mu_x x} \, e^{i(\mu_y y + \mu_z z - \omega t)} \\ E_{oy}^e &= A_{oy}^e \, e^{-\mu_x x} \, e^{i(\mu_y y + \mu_z z - \omega t)} \\ E_{oz}^e &= A_{oz}^e \, e^{-\mu_x x} \, e^{i(\mu_y y + \mu_z z - \omega t)} \end{aligned} \qquad (155)$$

Matching these to the interior solutions at x = d results in

$$A^e_{ox} = \tfrac{1}{2}A^i_{ox}\ (e^{2\mu_x d} + 1)$$

$$A^e_{oy} = \tfrac{1}{2}A^i_{oy}\ (e^{2\mu_x d} - 1) \tag{156}$$

$$A^e_{oz} = \tfrac{1}{2}A^i_{oy}\ (e^{2\mu_x d} - 1)$$

The conditions $\underline{\nabla}\cdot\underline{E}^{e,i}_o = 0$ gives

$$-\mu_x E^{e,i}_{ox} + i\mu_y\ E^{e,i}_{oy} + i\mu_z\ E^{e,i}_{oz} = 0 \tag{157}$$

Substituting Eq. (156) into (157), we obtain

$$-\mu_x A^i_{ox}\ (e^{2\mu_x d} + 1) = -\mu_x A^i_{ox}\ (e^{2\mu_x d} - 1).$$

Hence, $E^i_{ox} = E^e_{ox} = 0$, as expected, since $\underline{K}_o$ had no x component. This simplicity is lost in cylindrical geometry. As in Eq. (136), the $[B_{tan}]$ condition gives

$$i(E^e_{oy} - E^i_{oy})' = 4\pi K_{oy}/c \tag{158}$$

The middle equations in (154-156) then yield

$$A^i_{oy} = \frac{4\pi i}{c\mu_x}\ e^{-\mu_x d}\ K_{oy} \tag{159}$$

or $$E^i_{oy} = \frac{4\pi i}{c\mu_x}\ K_{oy}\ e^{-\mu_x d}\ \sinh \mu_x x, \tag{160}$$

and similarly with E^i_{oz}.

The enhancement factor Q for the case $b\to\infty$ is given by the ratio of Eqs. (151) and (160):

$$Q = \frac{E^i_y}{E^i_{oy}} = \left(1 + \frac{\mu_y^2}{\mu_z^2}\right) \frac{\mu_x\ e^{\mu_x a}}{\mu_x\ \sinh ga + g\ \cosh ga}\ \frac{\sinh gx.}{\sinh \mu_x x} \tag{161}$$

Note that the position d of the current sheet does not matter. Here g satisfies the dispersion relation

$$g^2 = \mu_y^2 + \mu_z^2 + \frac{S\mu_z^2 - RL}{S - \mu_z^2} \tag{162}$$

When ω is not near cyclotron resonance, the values of S, R, and L are usually much smaller than $\mu_z 2$. Hence, we may take the low-density limit $S = R = L = 1$, or $g = \mu_x$. In that case, Eq. (161) yields simply

$$Q = 1 + \frac{\mu_y 2}{\mu_x 2} , \tag{163}$$

which is exactly the value found previously [Eq. (110)] by neglecting the transverse field. Note that we can take the low-density limit without getting $Q = 1$ because E_y^i does not converge to E_{oy}^i under the infinite-conductivity assumption.

At ion cyclotron resonance, L goes to ∞ and S to L/2, while R remains finite. Then g^2 becomes

$$g^2 = \mu_y^2 + \mu_z^2 + \frac{L\mu_z^2 - 2RL}{L - 2\mu_z^2} \approx \mu_y^2 + 2\mu_z^2 > \mu_x^2 \tag{164}$$

One sees from Eq. (161) that Q remains finite even at $\omega = \Omega_c$, and in fact has a value that differs from Eq. (163) only by a factor of order unity. Thus, the intuitive answer is verified by exact calculation.

Besides E_y^i, there is also a component E_x^i that did not exist in vacuum; this is given by Eq. (149) or, for $b \to \infty$, by Eq. (152). Combining this with E_y^i according to Eq. (85), we can obtain the circularly polarized components:

$$E^{L,R} = -\frac{2^{3/2}\pi}{c} K_o e^{-\mu_x(d-a)} \frac{\mu_y(\sinh ga - g\mu_x \cosh ga)\cosh gx \pm \mu_x(\mu_y^2+\mu_z^2)\cosh ga \sinh gx}{\mu_x\mu_y\mu_z^2 \cosh ga\,(\mu_x \sinh ga + g\cosh ga)} \quad (165)$$

For E in V/cm and K_o in A/cm, the numerical coefficient should be replaced by $60\pi\sqrt{2}$. The vacuum solution from Eq. (160) is

$$\sqrt{2}\, E_o^{L,R} = \pm i E_{oy}^i = \mp \frac{4\pi}{c} \frac{K_o}{\mu_x\mu_y} e^{-\mu_x d} \sinh \mu_x x . \quad (166)$$

Dividing this, we find the enhancement factor to be

$$Q^{L,R} = \frac{e^{\mu_x a}}{\mu_z^2 \cosh ga\,(\mu_x \sinh ga + g\cosh ga)} \Big[\mu_x(\mu_y^2+\mu_z^2)\cosh ga \frac{\sinh gx}{\sinh \mu_x x} \pm \mu_y(\sinh ga - g\mu_x \cosh ga)\frac{\cosh gx}{\sinh \mu_x x}\Big] . \quad (167)$$

The first term is due to E_y; the second, to E_x. Since these have different x-dependences, they cannot cancel everywhere; and there is always field enhancement. However, for large gx a large degree of cancellation does occur for the L component, and one has to analyze this case more carefully.

To do this, let us simplify the problem by taking the low-density limit $g^2 \approx \mu_x^2 = \mu_y^2 + \mu_z^2 - 1$. From the dispersion relation

$$g^2 = \mu_y^2 + \mu_z^2 + \frac{S\mu_z^2 - RL}{S - \mu_z^2} \quad (162)$$

we see that the approximation $g^2 = \mu_x^2$ requires

$$|RL/S| << \mu_z^2 \quad \text{and} \quad |S| << \mu_z^2 . \quad (168)$$

The low frequency approximation $\omega^2 << \omega_c^2$ puts S, R, L into the form [cf. Eq. (121)]

$$S = 1 + \sum_s \frac{\Omega_p^2}{\Omega_c^2 - \omega^2} , \qquad R,L = 1 + \sum_s \frac{\Omega_p^2/\Omega_c}{\Omega_c \pm \omega} , \quad (169)$$

where the sum is over ion species. Let us choose a set of parameters to see when the conditions (168) are satisfied. We take

$$B = 20\text{kG},\quad M = 238M_H,\quad Z = 2,\quad n = 10^{12}\text{cm}^{-3},$$

$$a = 10\text{cm},\quad \lambda_y = 20\pi\text{cm},\quad \lambda_z = 60\pi\text{cm},\quad k_y/k_z = 3$$

From this we obtain

$$\begin{aligned}
&\Omega_c = 1.61 \times 10^6\ \text{sec}^{-1}, && f_c = 2.56 \times 10^5\ \text{Hz} \\
&k_o = 5.44 \times 10^{-5}\ \text{cm}^{-1}, && k_o a = 5.44 \times 10^{-4} \\
&\mu_y = 1.84 \times 10^3, && \mu_y^2 = 3.39 \times 10^6 \\
&\mu_z = 6.13 \times 10^2, && \mu_z^2 = 3.76 \times 10^5 \\
&\mu_x = 1.94 \times 10^3, && \mu_x^2 = 3.76 \times 10^6 \\
&k_x a = 1.055, && \\
&\Omega_p = 1.71 \times 10^8\ \text{sec}^{-1}, && \Omega_p^2 = 2.91 \times 10^{16}\ \text{sec}^{-2}
\end{aligned} \tag{170}$$

When ω is off resonance ($\omega^2 << \Omega_c{}^2$, say) then Eq. (169) gives for one species

$$S \approx R \approx L \approx 1.1 \times 10^4.$$

Hence $|RL/S| \approx |S| \approx 10^4 << \mu_z{}^2 = 3.8 \times 10^5$, and Eq. (168) is satisfied. However, S and L can become large near cyclotron resonance, and there will be a range of frequencies where the low-density approximation $g^2 = \mu_x{}^2$ fails.

To study the behavior of g near $\omega = \Omega_c$, define

$$\begin{aligned}
&A \equiv 1 + (k_y^2/k_z^2) && (=10 \text{ in this example}) \\
&\gamma \equiv \Omega_p^2/\Omega_c^2 && (\approx 10^4 \text{ in this example}) \\
&\Delta \equiv (\omega - \Omega_c)/\Omega_c &&
\end{aligned} \tag{171}$$

With the use of Eq. (169), Eq. (162) becomes

$$g^2 = A\mu_z^2 + \frac{\mu_z^2 - \gamma}{1 - (\mu_z^2/\Omega_p^2)(\Omega_c^2 - \omega^2)} \approx \frac{\mu_z^2\ (A + 1 + 2A\mu_z^2\Delta/\gamma) - \gamma}{1 + 2\mu_z^2\ \Delta/\gamma} \tag{172}$$

where we have replaced $\omega+\Omega_c$ by $2\Omega_c$. The last γ in the numerator shifts the zero only slightly, since γ is $\approx$ 300 times smaller than $A\mu_z^2$. We see that g^2 goes through ∞ at $\Delta = -\frac{1}{2}\gamma/\mu_z^2 = -.015$, and through 0 when Δ is $(1+A)/A = 1.1$ times this. Far from resonance, the Δ terms dominate, and $g^2 \simeq A\mu_z^2 = 10\mu_z^2$. At exact resonance, $\Delta = 0$ and $g^2 \simeq (1+A)\mu_z^2 = 11\mu_z^2$. Thus the behavior of $g^2(\omega)$ is as shown in Fig. 10.

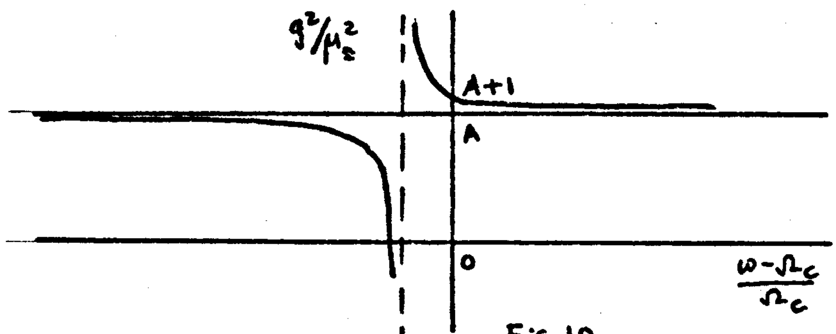

Fig. 10

Since the internal field E^i varies as $e^{\pm gx}$, positive g^2 means that the wave is damped as it propagates inward. Negative g^2 means that an undamped propagating wave is excited, but this can happen only between $\Delta = -1.493 \times 10^{-2}$ and -1.642×10^{-2} in this example. The width of the propagating region is only $\gamma/2A\mu_z^2 = 0.15\%$ of Ω_c. Since B is not usually uniform to this accuracy, there is to all intents and purposes no excitation of non-evanescent waves. Except in this narrow zone of ω, g^2 can be approximated by $A\mu_z^2 = \mu_y^2 + \mu_z^2 = \mu_x^2 + 1 \simeq \mu_x^2$. Thus the low-density approximation is almost always valid.

Now let us add a second ion species, say with $M=235M_H$, $n=10^{10}$ cm^{-3} and $Z=2$. There will be two terms in the sums of Eq. (169), but the resonances will be separated, since the Ω_c's differ by 1.3% while the resonance zone is only $\simeq$ 0.3% wide. For the minor species n is 100 times less, so that γ is also 100 times less. The resonance region is then only $\simeq 3 \times 10^{-5}\Omega_c$ wide and can be completely neglected. We have computed $g^2(\omega)$ for this two-ion case, but the curve need not be reproduced because it is just as described above.

Having justified the low-density approximation, we may set $g=\mu_x$ in Eq. (165). For simplicity we also omit the factor $e^{-\mu_x(d-a)}$, assuming that the plasma extends up to the excitation coil. We then get

$$E^{L,R}\left(\frac{V}{cm}\right) = 60\pi\sqrt{2}\, K_o\left(\frac{A}{cm}\right)\frac{e^{-\mu_x a}}{\mu_z^2}\left[\left(1-\frac{\tanh\mu_y a}{\mu_x^2}\right)\cosh\mu_x x \mp \frac{\mu_z^2}{\mu_x\mu_y}\left(1+\frac{\mu_y^2}{\mu_z^2}\right)\sinh\mu_x x\right]. \quad (173)$$

Dividing by the vacuum field of Eq. (166), we get for the enhancement factor

$$Q^{L,R} = \mp\frac{\mu_x\mu_y}{\mu_z^2}\left(1-\frac{\tanh\mu_y a}{\mu_x^2}\right)\mathrm{ctnh}\,\mu_x x + \left(1+\frac{\mu_y^2}{\mu_z^2}\right). \quad (174)$$

In these formulas, it is understood (because of the way the x-dependences were defined) that g and μ_x are positive, while μ_y and μ_z can take both positive and negative values. Q becomes infinite at x=0 because E_o vanishes while E_x^i does not.

The last term in Eq. (174) comes from E_y and is simply the factor A we obtained earlier in a cruder treatment [Eq. (110)]. The first term in Q comes from E_x and can cancel part of most of the last term. Near x=a, we have $\mu_x x \approx 1$ (x is really $k_o x$ here) since $k_x a = 1.055$ according to Eq. (170). In that case, ctnh $\mu_x x \approx \tanh \mu_x a \approx 1$. Q then becomes (since $\mu_x^2 \gg 1$)

$$Q^{L,R} \simeq \mp\frac{\mu_x\mu_y}{\mu_z^2}(\pm 1) + \left(1+\frac{\mu_y^2}{\mu_z^2}\right),$$

the ± 1 meaning positive or negative x. Since $\mu_x^2 \approx \mu_y^2$ if μ_z^2 is comparatively small, we see that $Q^L \approx 1$ for x > o, and $Q^R \approx 1$ for x < o. A more careful treatment shows that Q is ≈½ on one side of the plasma and is > A on the other, the left- and right-handed components peaking on opposite sides.

This peculiar behavior is removed if we remember that the real coil is represented by the sum of four exponentials with positive and negative choices for μ_y and μ_z. So we must add four solutions of the form (173), varying the signs of μ_y, μ_z. The first term of Eq. (173) does not depend on either sign, so the sum will yield a $\cos|k_y|y \cos|k_z|z$ dependence. The second term depends only on the sign of μ_y, so the sum will yield $i \cos|k_z|z \sin|k_y|y$. Now it is clear that the two terms in $E^{L,R}$ cannot cancel because they are shifted by 90° in y. Also, there is a factor "i" between the (E_x, E_y) components of (E^L, E^R) which means that the wave is not circularly polarized at all -- it is linearly polarized. So we may as well go back to the expressions for E_x and E_y.

For $b \to \infty$, $d = a$, and $g = \mu_x$, Eqs. (151) and (152) become

$$E_y^i = \frac{4\pi i}{c} \frac{K_o}{\mu_x \mu_y} \left(1 + \frac{\mu_y^2}{\mu_z^2}\right) e^{-\mu_x a} \sinh \mu_x x \tag{175}$$

$$\begin{aligned} E_x^i &= -\frac{4\pi}{c} \frac{K_o}{\mu_z^2} \frac{\cosh \mu_x x}{\cosh \mu_x a} \; \frac{1 - \mu_x^2 \operatorname{ctnh} \mu_x a}{\mu_x^2 (1 + \operatorname{ctnh} \mu_x a)} \\ &= \frac{4\pi}{c} \frac{K_o}{\mu_z^2} e^{-\mu_x a} \left(1 - \frac{\tanh \mu_x a}{\mu_x^2}\right) \cosh \mu_x x . \end{aligned} \tag{176}$$

The vacuum field is given by Eq. (160):

$$E_y^o = \frac{4\pi i}{c} \frac{K_o}{\mu_x \mu_y} e^{-\mu_x a} \sinh \mu_x x . \tag{177}$$

Now if we add four waves with $\pm k_y$, $\pm k_z$, the exponential factors will give $4i \sin|k_y|y \cos|k_z|z$ for Eq. (175), where the sign changes with μ_y, and will give $4 \cos|k_y|y \cos|k_z|z$ for Eq. (176), where there is no dependence on sign. In any case, it is clear that E_y^i is always A times E_y^o, while E_x^i/E_y^o varies with position. We obtain

$$E_x^i = \frac{4\pi i}{c} \frac{K_o'}{\mu_z^2} e^{-\mu_x a} \left(\frac{\tanh \mu_x a}{\mu_x^2} - 1\right) \cosh \mu_x x \cos|k_y|y \cos|k_z|z \tag{178}$$

$$E_y^i = \frac{4\pi i}{c} \frac{K_o'}{|\mu_x \mu_y|} e^{-\mu_x a} \left(1 + \frac{\mu_y^2}{\mu_z^2}\right) \sinh \mu_x x \sin|k_y|y \cos|k_z|z , \tag{179}$$

where $K' = 4iK_o$. We see that E_x and E_y are in phase, so that the wave is linearly polarized. The enhancement factors are

$$-\frac{E_x^i}{E_y^o} = \left|\frac{\mu_x\mu_y}{\mu_z^2}\right| \left(1 - \frac{\tanh \mu_x a}{\mu_x^2}\right) \operatorname{ctnh} \mu_x x \ \operatorname{ctn} |k_y| y \tag{180}$$

$$\frac{E_y^i}{E_y^o} = 1 + \frac{\mu_y^2}{\mu_z^2} . \tag{181}$$

The x-component is finite (in fact, maximum)) at x = o and so has an infinite enhancement factor there. Near the edge, where $\mu_x x \gtrsim 1$ so that $\operatorname{ctnh} \mu_x x \approx \tanh \mu_x a \approx 1$, we have

$$\frac{E_x^i}{E_y^o} \simeq \left|\frac{\mu_x\mu_y}{\mu_z^2}\right| \operatorname{ctn} k_y y .$$

The factor $|\mu_x\mu_y/\mu_z^2|$ is

$$\frac{\mu_x\mu_y}{\mu_z^2} = \frac{\mu_y}{\mu_z^2}\left(\mu_y^2+\mu_z^2-1\right)^{1/2} = \frac{\mu_y}{\mu_z}\left(\frac{\mu_y^2}{\mu_z^2}+1-\frac{1}{\mu_z^2}\right)^{1/2} \simeq \frac{\mu_y}{\mu_z}\left(1+\frac{\mu_y^2}{\mu_z^2}\right)^{1/2} \simeq 1+\frac{\mu_y^2}{\mu_z^2} .$$

Thus the E_x and E_y components have about the same amplification factor but are shifted in the y-direction.

The formulas (178) and (179) describe a linearly polarized field which is primarily the electrostatic field due to the space charges that build up to oppose E_z^O, plus a small correction due to the induced electromagnetic field. There is no distinction between left- and right-hand circularly polarized components as far as how well they penetrate into the plasma is concerned. This result holds at exact cyclotron resonance and fails only in the negligibly narrow frequency region where $g \to 0$. The infinite plasma theory, where the plasma response to a <u>given</u> internal field is calculated, does <u>not</u> give a finite total field at exact cyclotron resonance.

Finally, we wish to comment on the effect of the phasing of the K_{oy} and K_{oz} currents on the two excitation plates. In particular, one wonders whether a $\nabla \cdot K_o \neq 0$ arrangement with antisymmetric K_{oz} to mock up the situation in a cylinder would greatly affect the enhancement factor. Clearly not, since E_x and E_y are primarily due to a quasistatic distribution of space charges. On the other hand, the phasing would make a big difference if we had propagating waves of the form exp $i(k_y y + k_z z - \omega t)$, as we saw from the cancellation that can occur. The difference is that the "coil" we chose was excited at a frequency much lower than its natural frequency, so that L and R components were excited equally. If we had chosen $|\mu|$ closer to unity, it would have been possible to excite the L and R wave preferentially, especially if the coil were helical. Another peculiarity of the geometry treated here is that E_x^i and E_y^i both depend on cos $|k_z|z$, as seen in Eqs. (178) and (179). This means that $\underline{E}^i$ vanishes altogether at some positions z. In a helical coil, the components E_r and E_θ are out of phase, so that the total $\underline{E}^i$ has approximately constant amplitude along z. A particle moving along z, however, will see a Doppler shift as E_r and E_θ alternately become dominant.

The radial variation of E_x^i, E_y^i and $|E|$ in the low-density limit for the parameters of Eq. (170) is compared with E_y^0 in Fig. 11, showing the variation of the enhancement factor with radius. Here the y and z dependences have been neglected; in practice, the enhancement is lower by $\sqrt{2}$ because E_x and E_y do not peak at the same value of y.

The validity of the low-density approximation is illustrated in Fig. 12, where we plot $Q = (Q_x{}^2 + Q_y{}^2)^{1/2}$ vs. density n_1 of the major species, allowing g to vary from μ_x (depending on n_1) and fixing the position at x=a, tan $|k_y|y = 1$. The value of Q_y is given by Eq. (161), and Q_x is given by the corresponding equation

$$Q_x = \left|\frac{\mu_y}{\mu_x \mu_z^2}\right| e^{\mu_x a} \, \frac{\tanh ga - g\mu_x}{\tanh ga + g/\mu_x} \, \frac{\cosh gx}{\cosh ga} \, \frac{\tan |k_y| y}{\sinh \mu_x x} \quad . \tag{182}$$

In Fig. 12 it is further assumed that there are two ion species, with $n_2/n_1 = 0.1$, $M_2/M_1 = 235/238$, and $\omega = \Omega_2$ exactly. The expression for $g(n_1)$ analogous to Eq. (172) is then

$$g^2 = \mu_z^2 (A+1) - \left(1 + \frac{n_2}{n_1}\right) \frac{\Omega_{p2}^2}{\Omega_{c2}^2} \quad . \tag{183}$$

It is seen that Q is very close to the low-density value until g^2 goes negative, whereupon Q abruptly goes to ∞.

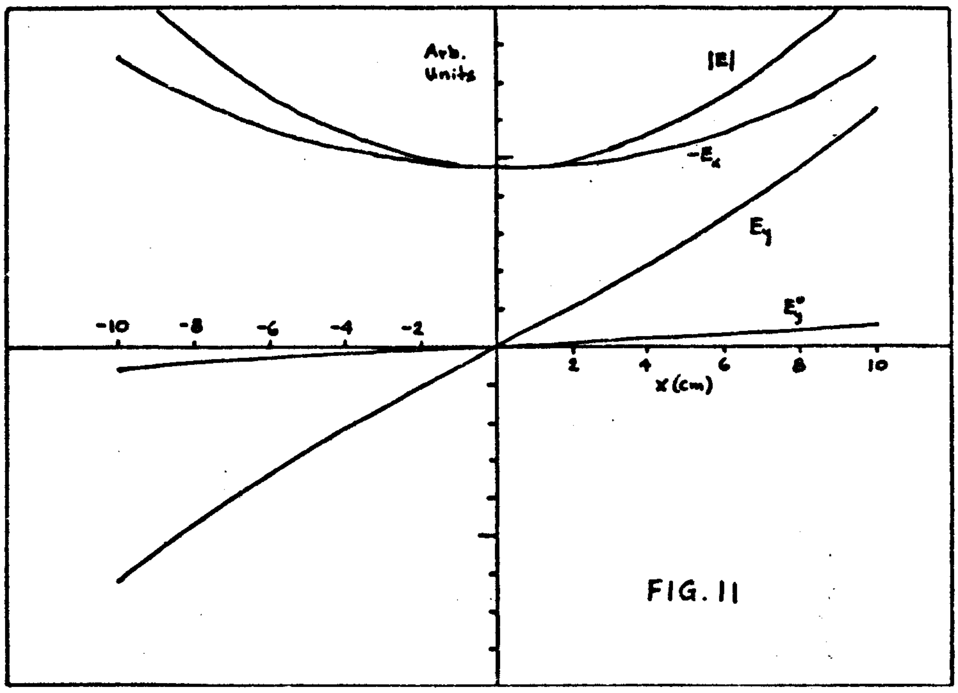
Arb. Units
|E|
$-E_x$
E_y
E_y^0
-10
-8
-6
-4
-2
2
4
6
8
10
x (cm)

FIG. 11

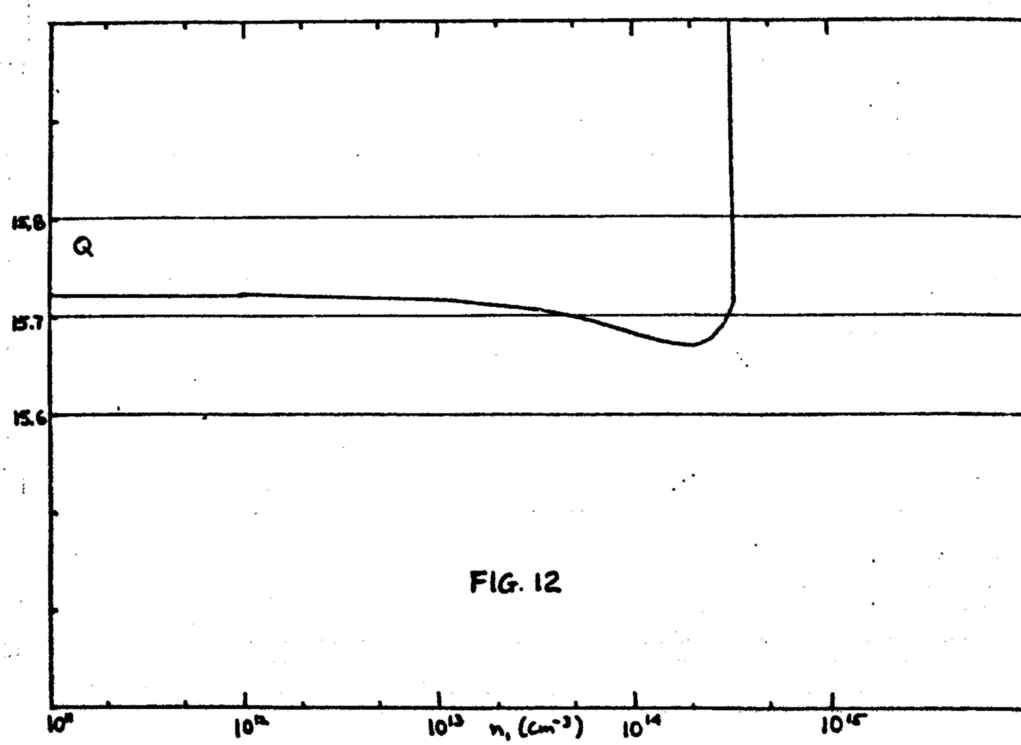
15.8
Q
15.7
15.6
10^{11}
10^{12}
10^{13}
n_1 (cm^{-3})
10^{14}
10^{15}

FIG. 12

8. SUMMARY

We have obtained analytic expressions for the rf field inside the plasma when excited by a Nagoya Type III coil in plane geometry. If the plasma response to a sinusoidal-in-x drive field is calculated, one gets approximately the right field enhancement factor, but its variation with frequency and position is misleading. In particular, this method predicts zero internal field at $\omega=\Omega_c$. If the plasma response to the actual vacuum field of the coil is calculated assuming that this drive field is unaffected by the plasma, then a non-trivial answer cannot be obtained, since we are dealing with a boundary condition problem rather than an infinite plasma. However, if only the electrostatic plasma response is calculated, a finite answer can be obtained, again with approximately the right enhancement factor [Eq. (110)]. We show that, to obtain an exact answer, the volume-excitation problem must be replaced by the usual surface-excitation problem, which is equivalent.

The resulting internal field is given by Eq. (165) for the case where the conducting boundary is at infinity; more complicated formulas for $b \neq \infty$ are given in Eqs. (148) and (149). Particularly simple formulas are available in the low-density limit, in which the plasma parameters do not appear at all. The region of validity of this limit does depend on the plasma, but we show that the approximation is always valid under the conditions of interest. There is then no difference between penetration of the L and R components. The plane-polarized components in this limit are given by Eqs. (178) and (179). The enhancement factor in the low-density limit is purely geometrical:

$$Q \approx 1 + k_y^2/k_z^2.$$

In particular, it retains approximately this value at $\omega=\Omega_c$.

This analytic treatment was possible because a cold, uniform plasma was assumed. Furthermore, the plasma was assumed to have perfect conductivity along the magnetic field. This last simplification can be removed by allowing E_{zz} to have a finite value depending on the thermal velocity and collision frequency in the fluid model, or on the Z' function in the kinetic model. The 2 x 2 matrices then become 3 x 3 matrices. A plasma which is warm also in the direction $\perp$ to B will have much more complicated perpendicular elements of $\underline{\underline{\varepsilon}}$. Whenever the temperature is finite, $\underline{\underline{\varepsilon}}$ will depend on $\underline{k}$, thus making it hard to solve for $\underline{k}$. However, k_y and and k_z are fixed in this treatment, and k_x can be given an approximate value in the small thermal correction terms.

More complicated effects such as inhomogeneity of the plasma and the range of k's excited by a finite coil are best treated by computation. Our results seem to agree with such computations where a comparison can be made.

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