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BRIEF COMMUNICATION

A non-singular helicon wave equation for a non-uniform plasma

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Abstract. A second-order differential equation for the r component of the magnetic field of a helicon wave in a non-uniform cylindrical plasma has been derived. The equation shows no singularities and can be integrated numerically to obtain the radial dependence of the electric and magnetic field components of the helicon wave. Eigenvalues and field profiles have been calculated for a parabolic density profile with the 'matrix shooting' method.

In a recent paper Chen *et al* [1] derived a dispersion relation for helicon waves in a cold, cylindrical plasma with an arbitrary radial density profile. The wave equation they obtained contained coefficients which became infinite at a specific radius. Though this singularity has no physical significance, it has to be treated carefully in numerical solutions. In this brief communication we show that even though the equation for B_z is singular, the solutions themselves are well behaved. We give an alternative wave equation that has no singularity and, therefore, is more suitable for numerical treatment. The equation is for the r component of the magnetic field and has been solved to yield both the radial eigenvalue and the eigenmode. A 'matrix shooting' method was used; the eigenvalues and eigenmodes obtained agree with the previous results.

The equation reported [1] can be expressed as

$$B_z'' + \frac{F(r)}{\beta(r)} B_z' + \frac{G(r)}{\beta(r)} B_z = 0$$

where $F(r)$ and $G(r)$ are both analytic, and the denominator $\beta(r)$ (defined by Chen *et al* [1]) is equal to zero at a radius r_0. It can be shown that r_0 is a regular singular point by expanding $\beta(r)$ about it. Following the method of Frobenius [2] we assume a modified power series solution for B_z of the form $B_z = (r - r_0)^s \Sigma_{n=0}^{\infty} a_n (r - r_0)^n$ and obtain the indicial equation $s(s-2) = 0$, with solutions $s_1 = 2$ and $s_2 = 0$. The two independent solutions B_{z1} and B_{z2} have the general forms:

$$B_{z1} = (r - r_0)^2 \sum_{n=0}^{\infty} a_n (r - r_0)^n$$

$$B_{z2} = \sum_{n=0}^{\infty} b_n (r - r_0)^n + c B_{z1} \ln(r - r_0)$$

where the constant c may or may not be zero, depending on the recurrence formula for b_n. It is obvious from the above equations that B_{z1}, B_{z2}, and, therefore, B_z, are bounded at r_0. We therefore search for another equation which would not engender numerical difficulties.

The equation for the B_r component fulfills this requirement and can be derived as follows. The plasma is cylindrical and has an axial magnetic field $B_0\hat{z}$. The linearized Maxwell's equations are:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} - \mathrm{i}\frac{\omega}{c^2}\boldsymbol{E} \tag{1}$$

$$\nabla \times \boldsymbol{E} = \mathrm{i}\omega \boldsymbol{B} \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3}$$

The perpendicular electric field is given by

$$\boldsymbol{E} = \frac{\boldsymbol{J} \times \boldsymbol{B}_0}{ne} \tag{4}$$

where we have neglected E_z, ion motion, electron inertia, and any damping of the wave. We define the parameter α as

$$\alpha = \frac{\omega}{k}\frac{e\mu_0}{B_0} n_0(r) \tag{5}$$

and notice that it depends on radial position through the density. Using equation (2) together with the curl of equation (4) yields

$$\frac{\alpha}{\mu_0}\boldsymbol{B} = \boldsymbol{J} + \frac{\mathrm{i}}{k}\left(\nabla \cdot \boldsymbol{J} - \frac{\alpha'}{\alpha} J_r\right)\hat{z}. \tag{6}$$

Combining the curl of equations (1) and (4) we get

$$\nabla^2 \boldsymbol{B} + \mu \nabla \times \boldsymbol{J} + k_0^2 \boldsymbol{B} = 0 \tag{7}$$

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where $k_0 \equiv \omega/c$. The r component of this equation is

$$B_r'' + \frac{1}{r}B_r' - \left(\frac{m^2+1}{r^2} + k^2 - k_0^2\right)B_r - \frac{2im}{r^2}B_\theta + \mu_0\left(\frac{im}{r}J_z - ikJ_\theta\right) = 0 \quad (8)$$

where m is the azimuthal mode number and $(') = \partial/\partial r$. Equation (1), together with equation (4), gives j_z as a function of B_r and B_θ, while the r component of equation (6) give J_r as a function of B_r. With these substitutions we obtain

$$B_r'' + \frac{1}{r}B_r' - \left(\frac{1}{r^2} + k^2\gamma\right)B_r + \frac{im}{r}B_\theta' - \mathrm{i}\left(\frac{m}{r^2} + k\alpha\right)B_\theta = 0 \quad (9)$$

where $\gamma \equiv 1 - (k_0/k)^2$. The θ component of equation (6) gives J_θ as a function of B_θ. Using equation (4) for the electric field and expressing J_r and J_θ in terms of B_r and B_θ, we can write the r and θ components of equation (1) as:

$$\alpha B_r = \frac{im}{r}B_z - ik\gamma B_\theta \quad (10)$$

$$\alpha B_\theta = ik\gamma B_r - B_z'. \quad (11)$$

Equation (3) allows us to eliminate B_z from equation (10), yielding

$$B_\theta = \frac{imrB_r' + \mathrm{i}(m + \alpha kr^2)B_r}{m^2 + \gamma k^2 r^2} \quad (12)$$

Substituting this result in equation (3) yields

$$B_z = \frac{\mathrm{i}\gamma kr^2 B_r' + \mathrm{i}(\gamma kr - m\alpha r)B_r}{m^2 + \gamma k^2 r^2} \quad (13)$$

To obtain B_θ', we multiply equation (10) by r and differentiate, then substitute the resulting B_z' in equation (11). Using this result in equation (9) together with (12) gives, after simplification, the final differential equation:

$$B_r'' + \left(1 + \frac{2m^2}{m^2 + \gamma k^2 r^2}\right)\frac{B_r'}{r} - \left[m^2 + \frac{mr\alpha}{k\gamma}\frac{\alpha'}{\alpha} - \frac{r^2}{\gamma}(\alpha^2 - k^2\gamma^2) - \frac{m^2 + kr^2(2m\alpha - k\gamma)}{m^2 + \gamma k^2 r^2}\right]\frac{B_r}{r^2} = 0. \quad (14)$$

The equation can be integrated numerically for an arbitrary density profile $\alpha(r)$ subject to the boundary condition $B_r = 0$ at the outer edge of the plasma ($r = a$).

We can then solve for the other field components from equations (12) and (13).

We have computed the radial dependence of the magnetic field for a parabolic density profile, defined by:

$$\frac{n(r)}{n_0} = \frac{\alpha}{\alpha_0} = 1 - \left(\frac{r}{a}\right)^2.$$

The method employed is 'matrix shooting' [3]. The interval $[0, a]$ is divided in a grid of N points and equation (14) is expressed in matrix form:

$$\boldsymbol{C} \cdot \boldsymbol{X} = 0$$

where $\boldsymbol{X}^\dagger = [B_{r1}, B_{r2}, \ldots, B_{rN}]$ and $\boldsymbol{C}$ is a banded matrix that contains coefficients obtained with the method of finite differences. The boundary conditions allow us to write B_{r1} as a function of B_{r2} and rewrite the matrix equation as $\boldsymbol{D} \cdot \boldsymbol{Y} = \boldsymbol{Z}(B_{r2})$, where $\boldsymbol{Y}^\dagger = [B_{r2}, B_{r3}, \ldots, B_{rN}]$, and the value of B_r at the second grid point is an arbitrary constant. The eigenvalue problem is solved by shooting the matrix, varying the eigenvalue α_0 until B_{rN} matches the boundary condition at $r = a$.

The boundary conditions for $m = 0$ are $B_r(0) = 0$ and $B_r(a) = 0$, i.e. $J_r = 0$. For $m \neq 0$ $B_r(a)$ is also 0; and in the case $m = \pm 1$, the second boundary condition is $B_r'(0) = 0$. This result is obtained from equation (14) in the following manner. We assume that in the limit as r approaches 0, B_r is proportional to r^n, where n is a real number. Equation (14) in the limit of small r is then equivalent to

$$n^2 + 2n - (m^2 - 1) = 0$$

with solutions $n = -1 \pm |m|$. The solution r^{-2} is not physical. Hence, B_r is a constant at small r and $B_r'(0) = 0$. The wave patterns calculated from equation (14) together with these boundary conditions agree with those computed for the B_z equation [1].

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