3. Matrices

- notation and terminology
- matrix operations
- linear and affine functions
- complexity

Matrix

a rectangular array of numbers, for example

$$A = \begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix}$$

- numbers in array are the *elements* (*entries*, *coefficients*, *components*)
- A_{ij} is the i, j element of A; i is its *row index*, j the *column index*
- *size* (*dimensions*) of the matrix is specified as (#rows) $\times$ (#columns)

  for example, the matrix A above is a 3×4 matrix
- set of $m \times n$ matrices with real elements is written $\mathbf{R}^{m\times n}$
- set of $m \times n$ matrices with complex elements is written $\mathbf{C}^{m\times n}$

Other conventions

- many authors use parentheses as delimiters:

$$A = \begin{pmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{pmatrix}$$

- often a_{ij} is used to denote the i, j element of A

Matrix shapes

Scalar: we don't distinguish between a 1×1 matrix and a scalar

Vector: we don't distinguish between an $n \times 1$ matrix and an n-vector

Row and column vectors

- a $1 \times n$ matrix is called a *row vector*
- an $n \times 1$ matrix is called a *column vector* (or just *vector*)

Tall, wide, square matrices: an $m \times n$ matrix is

- *tall* if $m > n$
- *wide* if $m < n$
- *square* if $m = n$

Block matrix

- a *block matrix* is a rectangular array of matrices
- elements in the array are the *blocks* or *submatrices* of the block matrix

Example

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

is a 2×2 block matrix; if the blocks are

$$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 4 & 7 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 6 & 0 \end{bmatrix}$$

then

$$A = \begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 5 & 4 & 7 \\ 1 & -1 & 6 & 0 \end{bmatrix}$$

Note: dimensions of the blocks must be compatible

Rows and columns

a matrix can be viewed as a block matrix with row/column vector blocks

- $m \times n$ matrix A as $1 \times n$ block matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

each a_j is an m-vector (the jth *column* of A)

- $m \times n$ matrix A as $m \times 1$ block matrix

$$A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

each b_i is a $1 \times n$ row vector (the ith *row* of A)

Special matrices

Zero matrix

- matrix with $A_{ij} = 0$ for all i, j
- notation: 0 (usually) or $0_{m \times n}$ (if dimension is not clear from context)

Identity matrix

- square matrix with $A_{ij} = 1$ if $i = j$ and $A_{ij} = 0$ if $i \neq j$
- notation: I (usually) or I_n (if dimension is not clear from context)
- columns of I_n are unit vectors $e_1, e_2, \ldots, e_n$; for example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

Symmetric and Hermitian matrices

Symmetric matrix: square with $A_{ij} = A_{ji}$

$$\begin{bmatrix} 4 & 3 & -2 \\ 3 & -1 & 5 \\ -2 & 5 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 4+3\mathrm{j} & 3-2\mathrm{j} & 0 \\ 3-2\mathrm{j} & -\mathrm{j} & -2\mathrm{j} \\ 0 & -2\mathrm{j} & 3 \end{bmatrix}$$

Hermitian matrix: square with $A_{ij} = \bar{A}_{ji}$ (complex conjugate of A_{ij})

$$\begin{bmatrix} 4 & 3-2\mathrm{j} & -1+\mathrm{j} \\ 3+2\mathrm{j} & -1 & 2\mathrm{j} \\ -1-\mathrm{j} & -2\mathrm{j} & 3 \end{bmatrix}$$

note: diagonal elements are real (since $A_{ii} = \bar{A}_{ii}$)

Structured matrices

matrices with special patterns or structure arise in many applications

- diagonal matrix: square with $A_{ij} = 0$ for $i \neq j$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

- lower triangular matrix: square with $A_{ij} = 0$ for $i < j$

$$\begin{bmatrix} 4 & 0 & 0 \\ 3 & -1 & 0 \\ -1 & 5 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

- upper triangular matrix: square with $A_{ij} = 0$ for $i > j$

Sparse matrices

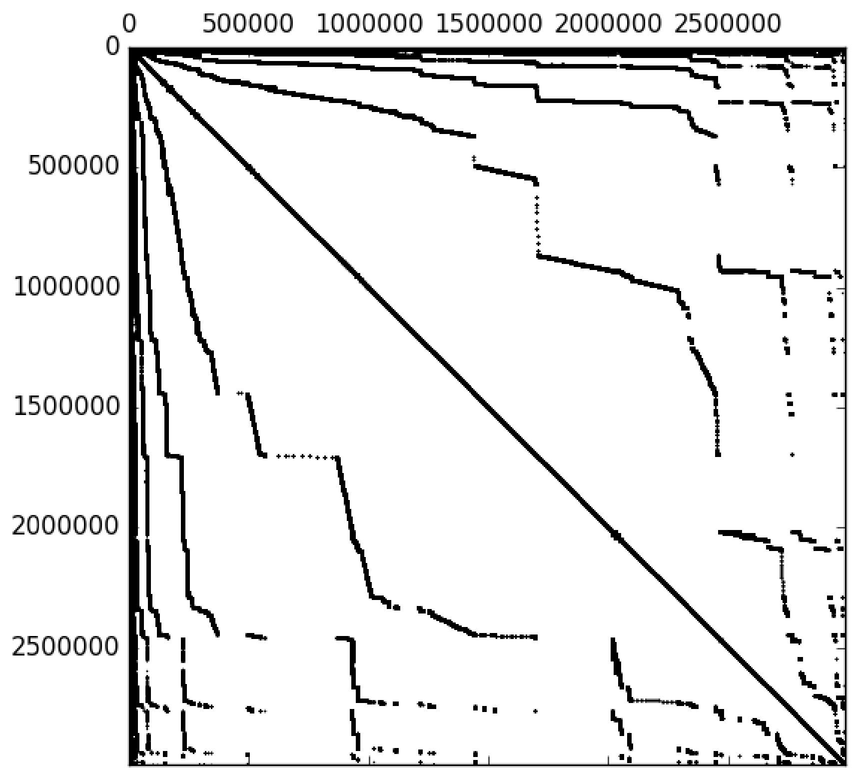
a matrix is *sparse* if most (almost all) of its elements are zero

- sparse matrix storage formats and algorithms exploit sparsity
- efficiency depends on number of nonzeros and their positions
- positions of nonzeros are visualized in a 'spy plot'

Example

- 2,987,012 rows and columns
- 26,621,983 nonzeros

(Freescale/FullChip matrix from
SuiteSparse Matrix Collection)



Outline

- notation and terminology
- **matrix operations**
- linear and affine functions
- complexity

Scalar–matrix multiplication and addition

Scalar–matrix multiplication:

scalar–matrix product of $m \times n$ matrix A with scalar β

$$\beta A = \begin{bmatrix} \beta A_{11} & \beta A_{12} & \cdots & \beta A_{1n} \\ \beta A_{21} & \beta A_{22} & \cdots & \beta A_{2n} \\ \vdots & \vdots & & \vdots \\ \beta A_{m1} & \beta A_{m2} & \cdots & \beta A_{mn} \end{bmatrix}$$

A and β can be real or complex

Addition: sum of two $m \times n$ matrices A and B (real or complex)

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2n} + B_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & \cdots & A_{mn} + B_{mn} \end{bmatrix}$$

Transpose

the *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix

$$A^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix}$$

- $(A^T)^T = A$
- a symmetric matrix satisfies $A = A^T$
- A may be complex, but transpose of a complex matrix is rarely needed
- transpose of scalar–matrix product and matrix sum

$$(\beta A)^T = \beta A^T, \qquad (A + B)^T = A^T + B^T$$

Conjugate transpose

the *conjugate transpose* of an $m \times n$ matrix A is the $n \times m$ matrix

$$A^H = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{21} & \cdots & \bar{A}_{m1} \\ \bar{A}_{12} & \bar{A}_{22} & \cdots & \bar{A}_{m2} \\ \vdots & \vdots & & \vdots \\ \bar{A}_{1n} & \bar{A}_{2n} & \cdots & \bar{A}_{mn} \end{bmatrix}$$

($\bar{A}_{ij}$ is complex conjugate of A_{ij})

- $A^H = A^T$ if A is a real matrix
- a Hermitian matrix satisfies $A = A^H$
- conjugate transpose of scalar–matrix product and matrix sum

$$(\beta A)^H = \bar{\beta} A^H, \qquad (A + B)^H = A^H + B^H$$

Matrix–matrix product

product of $m \times n$ matrix A and $n \times p$ matrix B (A, B are real or complex)

$$C = AB$$

is the $m \times p$ matrix with i, j element

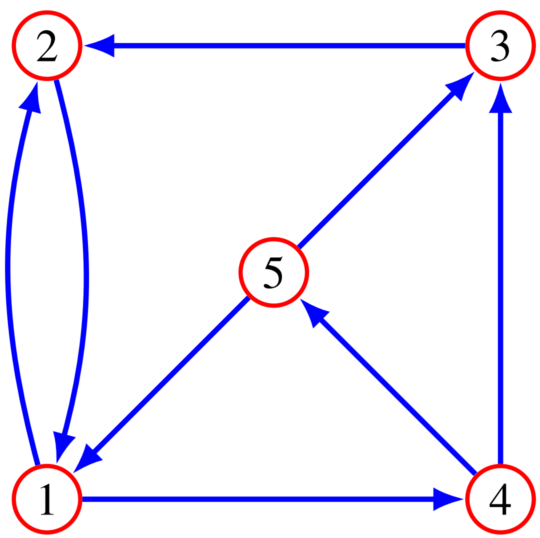
$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

dimensions must be compatible:

$$\text{\#columns in } A = \text{\#rows in } B$$

Exercise: paths in directed graph

directed graph with $n = 5$ vertices



matrix representation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$A_{ij} = 1$ indicates an edge $j \to i$

Question: give a graph interpretation of $A^2 = AA$, $A^3 = AAA$, ...

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad \ldots$$

Properties of matrix–matrix product

- *associative:* $(AB)C = A(BC)$ so we write ABC
- *associative with scalar–matrix multiplication:* $(\gamma A)B = \gamma(AB) = \gamma AB$
- *distributes with sum:*

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC$$

- *transpose and conjugate transpose of product:*

$$(AB)^T = B^T A^T, \qquad (AB)^H = B^H A^H$$

- ***not** commutative:* $AB \neq BA$ in general; for example,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

  there are exceptions, *e.g.*, $AI = IA$ for square A

Notation for vector inner product

- inner product of $a, b \in \mathbf{R}^n$ (see page 1.15):

$$b^T a = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}^T \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

  product of the transpose of the column vector b and the column vector a

- inner product of $a, b \in \mathbf{C}^n$ (see page 1.21):

$$b^H a = \bar{b}_1 a_1 + \bar{b}_2 a_2 + \cdots + \bar{b}_n a_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}^H \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

  product of conjugate transpose of column vector b and column vector a

Matrix–matrix product and block matrices

block-matrices can be multiplied as regular matrices

Example: product of two 2×2 block matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} W & Y \\ X & Z \end{bmatrix} = \begin{bmatrix} AW + BX & AY + BZ \\ CW + DX & CY + DZ \end{bmatrix}$$

if the dimensions of the blocks are compatible

Outline

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Matrix–vector product

product of $m \times n$ matrix A with n-vector (or $n \times 1$ matrix) x

$$Ax = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{bmatrix}$$

- dimensions must be compatible: number of columns of A equals the size of x
- Ax is a linear combination of the columns of A:

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

each a_i is an m-vector (ith column of A)

Linear function

a function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is **linear** if the superposition property

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

holds for all n-vectors x, y and all scalars α, β

Extension: if f is linear, superposition holds for any linear combination:

$$f(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_p u_p) = \alpha_1 f(u_1) + \alpha_2 f(u_2) + \cdots + \alpha_p f(u_p)$$

for all scalars, $\alpha_1, \ldots, \alpha_p$ and all n-vectors $u_1, \ldots, u_p$

Matrix–vector product function

for fixed $A \in \mathbf{R}^{m\times n}$, define a function $f : \mathbf{R}^n \to \mathbf{R}^m$ as

$$f(x) = Ax$$

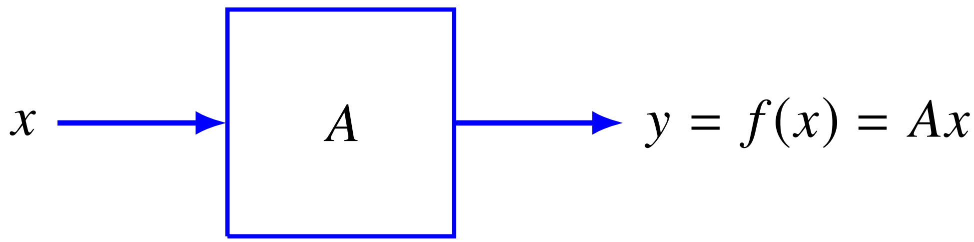
- any function of this type is linear: $A(\alpha x + \beta y) = \alpha(Ax) + \beta(Ay)$
- every linear function can be written as a matrix–vector product function:

$$\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n) \\ &= \begin{bmatrix} f(e_1) & \cdots & f(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

hence, $f(x) = Ax$ with $A = \begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix}$

Input–output (operator) interpretation

think of a function $f : \mathbf{R}^n \to \mathbf{R}^m$ in terms of its effect on x



- signal processing/control interpretation: n inputs x_i, m outputs y_i
- f is linear if we can represent its action on x as a product $f(x) = Ax$

Examples ($f : \mathbf{R}^3 \to \mathbf{R}^3$)

- f reverses the order of the components of x

  a linear function: $f(x) = Ax$ with

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- f sorts the components of x in decreasing order: not linear
- f scales x_1 by a given number d_1, x_2 by d_2, x_3 by d_3

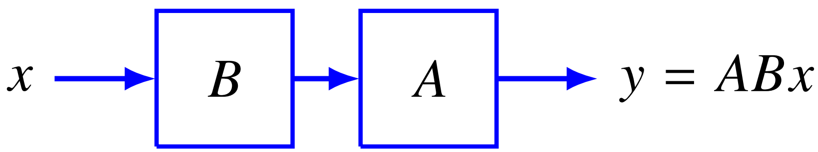
  a linear function: $f(x) = Ax$ with

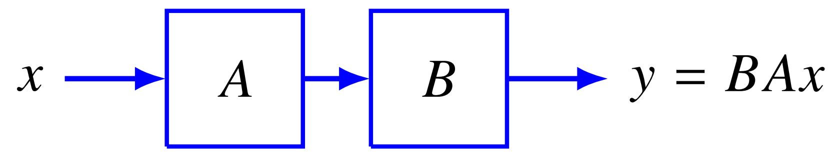
$$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

- f replaces each x_i by its absolute value $|x_i|$: not linear

Operator interpretation of matrix–matrix product

explains why in general $AB \neq BA$





Example

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- $f(x) = ABx$ reverses order of elements; then changes sign of first element
- $f(x) = BAx$ changes sign of 1st element; then reverses order

Reverser and circular shift

Reverser matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad Ax = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{bmatrix}$$

Circular shift matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad Ax = \begin{bmatrix} x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Permutation

Permutation matrix

- a square 0-1 matrix with one element 1 per row and one element 1 per column
- equivalently, an identity matrix with columns reordered
- equivalently, an identity matrix with rows reordered

Ax is a permutation of the elements of x

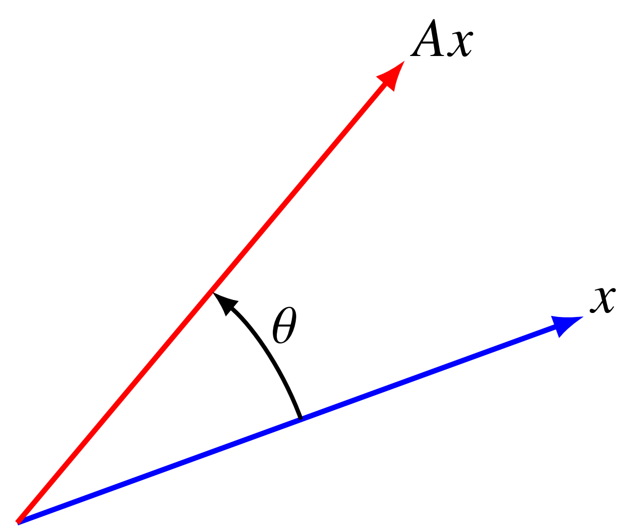
Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad Ax = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_3 \end{bmatrix}$$

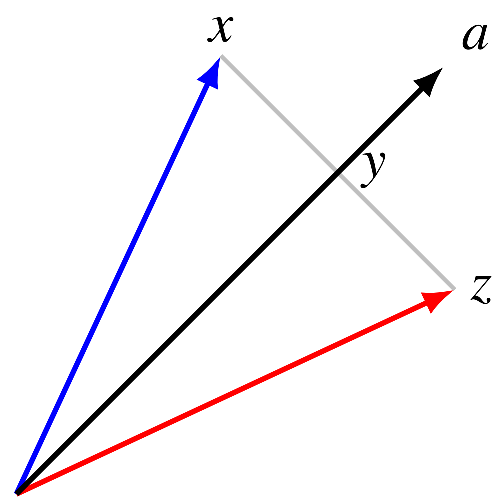
Rotation in a plane

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Ax is x rotated counterclockwise over an angle θ



Projection on line and reflection



- projection on line through a (see page 2.12):

$$y = \frac{a^T x}{\|a\|^2} a = Ax \quad \text{with} \quad A = \frac{1}{\|a\|^2} a a^T$$

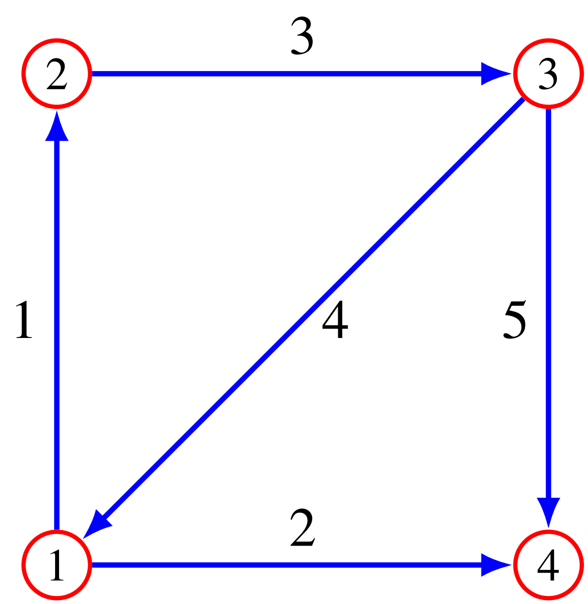
- reflection with respect to line through a

$$z = x + 2(y - x) = Bx, \quad \text{with} \quad B = \frac{2}{\|a\|^2} a a^T - I$$

Node–arc incidence matrix

- directed graph (network) with m vertices, n arcs (directed edges)
- incidence matrix is $m \times n$ matrix A with

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ enters node } i \\ -1 & \text{if arc } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

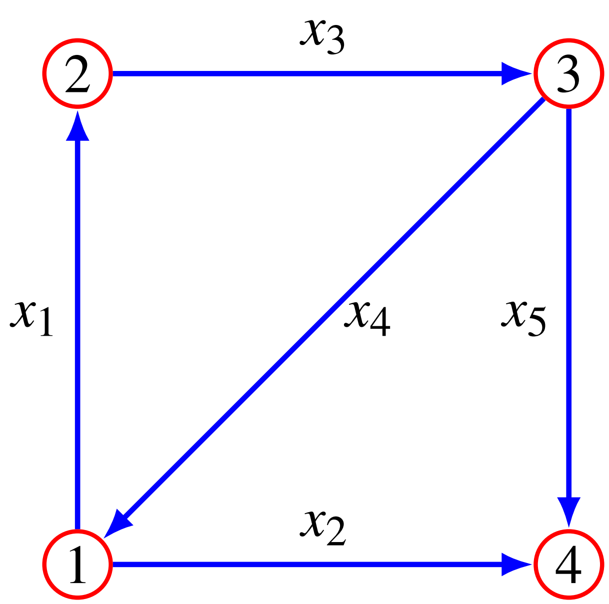


$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Kirchhoff's current law

n-vector $x = (x_1, x_2, \ldots, x_n)$ with x_j the current through arc j

$$
\begin{aligned}
(Ax)_i &= \sum_{\substack{\text{arc } j \text{ enters} \\ \text{node } i}} x_j - \sum_{\substack{\text{arc } j \text{ leaves} \\ \text{node } i}} x_j \\
&= \text{total current arriving at node } i
\end{aligned}
$$

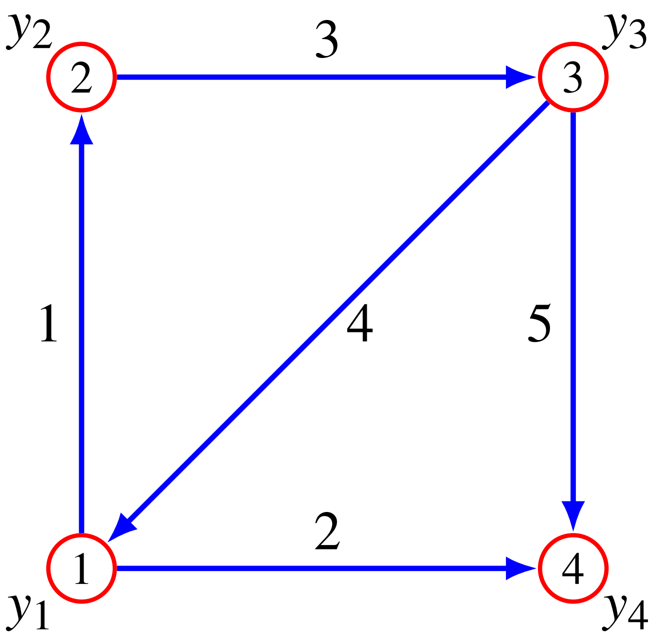


$$
Ax = \begin{bmatrix} -x_1 - x_2 + x_4 \\ x_1 - x_3 \\ x_3 - x_4 - x_5 \\ x_2 + x_5 \end{bmatrix}
$$

Kirchhoff's voltage law

m-vector $y = (y_1, y_2, \ldots, y_m)$ with y_i the potential at node i

$$\begin{aligned}(A^T y)_j &= y_k - y_l \quad \text{if edge } j \text{ goes from node } l \text{ to } k \\ &= \text{negative of voltage across arc } j\end{aligned}$$



$$A^T y = \begin{bmatrix} y_2 - y_1 \\ y_4 - y_1 \\ y_3 - y_2 \\ y_1 - y_3 \\ y_4 - y_3 \end{bmatrix}$$

Convolution

the *convolution* of an n-vector a and an m-vector b is the $(n+m-1)$-vector c

$$c_k = \sum_{\substack{\text{all } i \text{ and } j \text{ with} \\ i+j=k+1}} a_i b_j$$

notation: $c = a * b$

Example: $n = 4$, $m = 3$

$$\begin{aligned}
c_1 &= a_1 b_1 \\
c_2 &= a_1 b_2 + a_2 b_1 \\
c_3 &= a_1 b_3 + a_2 b_2 + a_3 b_1 \\
c_4 &= a_2 b_3 + a_3 b_2 + a_4 b_1 \\
c_5 &= a_3 b_3 + a_4 b_2 \\
c_6 &= a_4 b_3
\end{aligned}$$

Properties

Interpretation: if a and b are the coefficients of polynomials

$$p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}, \qquad q(x) = b_1 + b_2 x + \cdots + b_m x^{m-1}$$

then $c = a * b$ gives the coefficients of the product polynomial

$$p(x)q(x) = c_1 + c_2 x + c_3 x^2 + \cdots + c_{n+m-1} x^{n+m-2}$$

Properties

- symmetric: $a * b = b * a$
- associative: $(a * b) * c = a * (b * c)$
- if $a * b = 0$ then $a = 0$ or $b = 0$

these properties follow directly from the polynomial product interpretation

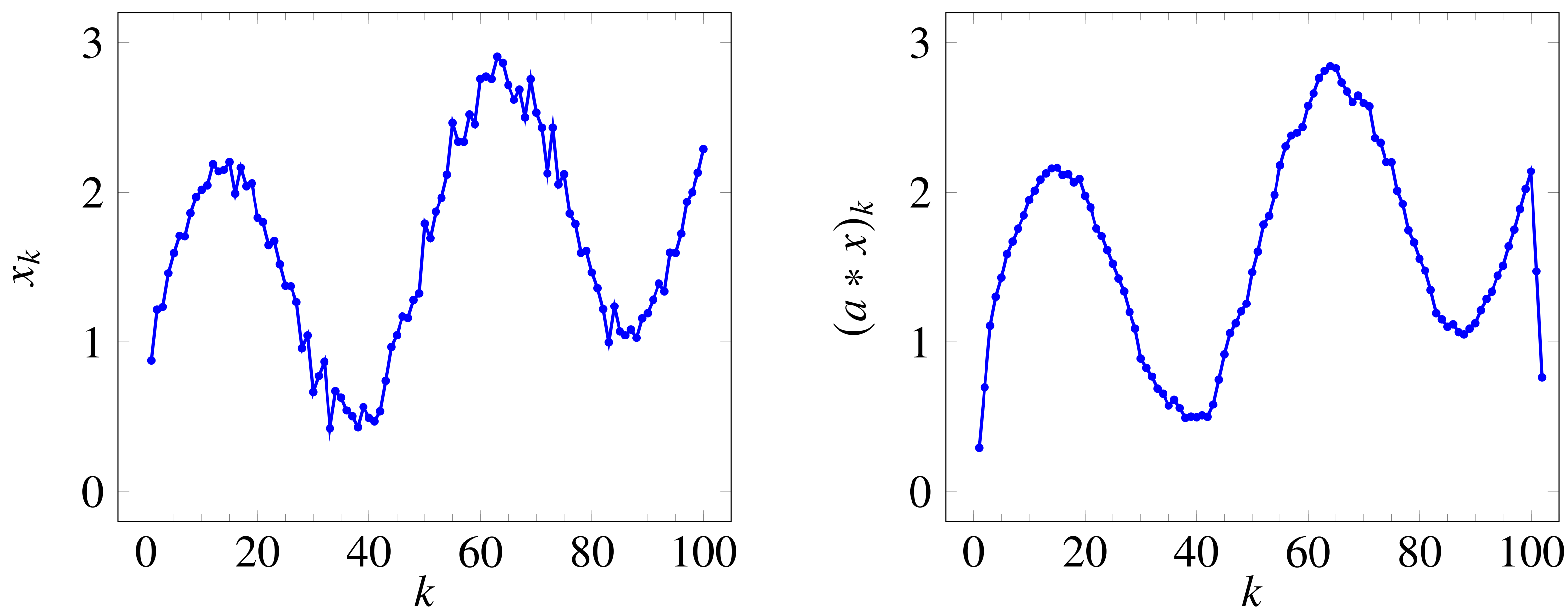
Example: moving average of a time series

- n-vector x represents a time series
- the 3-period *moving average* of the time series is the time series

$$y_k = \frac{1}{3}(x_k + x_{k-1} + x_{k-2}), \qquad k = 1, 2, \ldots, n+2$$

(with x_k interpreted as zero for $k < 1$ and $k > n$)

- this can be expressed as a convolution $y = a * x$ with $a = (1/3, 1/3, 1/3)$



Convolution and Toeplitz matrices

- $c = a * b$ is a linear function of b if we fix a
- $c = a * b$ is a linear function of a if we fix b

Example: convolution $c = a * b$ of a 4-vector a and a 3-vector b

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_1 & 0 \\ a_3 & a_2 & a_1 \\ a_4 & a_3 & a_2 \\ 0 & a_4 & a_3 \\ 0 & 0 & a_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ b_3 & b_2 & b_1 & 0 \\ 0 & b_3 & b_2 & b_1 \\ 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

the matrices in these matrix–vector products are called *Toeplitz* matrices

Vandermonde matrix

- polynomial of degree $n-1$ or less with coefficients $x_1, x_2, \ldots, x_n$:

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

- values of $p(t)$ at m points $t_1, \ldots, t_m$:

$$\begin{aligned} \begin{bmatrix} p(t_1) \\ p(t_2) \\ \vdots \\ p(t_m) \end{bmatrix} &= \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_m & \cdots & t_m^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= Ax \end{aligned}$$

the matrix A is called a *Vandermonde* matrix

- $f(x) = Ax$ maps coefficients of polynomial to function values

Discrete Fourier transform

the DFT maps a complex n-vector $(x_1, x_2, \ldots, x_n)$ to the complex n-vector

$$
\begin{aligned}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}
&= \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\
1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)(n-1)}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \\
&= Wx
\end{aligned}
$$

where $\omega = e^{2\pi \mathrm{j}/n}$ (and $\mathrm{j} = \sqrt{-1}$)

- DFT matrix $W \in \mathbf{C}^{n \times n}$ has k, l element $W_{kl} = \omega^{-(k-1)(l-1)}$
- a Vandermonde matrix with $m = n$ and

$$
t_1 = 1, \qquad t_2 = \omega^{-1}, \qquad t_3 = \omega^{-2}, \qquad \ldots, \qquad t_n = \omega^{-(n-1)}
$$

Affine function

a function $f : \mathbf{R}^n \to \mathbf{R}^m$ is **affine** if it satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all n-vectors x, y and all scalars α, β with $\alpha + \beta = 1$

Extension: if f is affine, then

$$f(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m) = \alpha_1 f(u_1) + \alpha_2 f(u_2) + \cdots + \alpha_m f(u_m)$$

for all n-vectors $u_1, \ldots, u_m$ and all scalars $\alpha_1, \ldots, \alpha_m$ with

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$$

Affine functions and matrix–vector product

for fixed $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, define a function $f : \mathbf{R}^n \to \mathbf{R}^m$ by

$$f(x) = Ax + b$$

i.e., a matrix–vector product plus a constant

- any function of this type is affine: if $\alpha + \beta = 1$ then

$$A(\alpha x + \beta y) + b = \alpha(Ax + b) + \beta(Ay + b)$$

- every affine function can be written as $f(x) = Ax + b$ with:

$$A = \begin{bmatrix} f(e_1) - f(0) & f(e_2) - f(0) & \cdots & f(e_n) - f(0) \end{bmatrix}$$

  and $b = f(0)$

Affine approximation

first-order Taylor approximation of differentiable $f : \mathbf{R}^n \to \mathbf{R}^m$ around z:

$$\widehat{f_i}(x) = f_i(z) + \frac{\partial f_i}{\partial x_1}(z)(x_1 - z_1) + \cdots + \frac{\partial f_i}{\partial x_n}(z)(x_n - z_n), \quad i = 1, \ldots, m$$

in matrix–vector notation: $\widehat{f}(x) = f(z) + Df(z)(x - z)$ where

$$Df(z) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(z) & \frac{\partial f_1}{\partial x_2}(z) & \cdots & \frac{\partial f_1}{\partial x_n}(z) \\ \frac{\partial f_2}{\partial x_1}(z) & \frac{\partial f_2}{\partial x_2}(z) & \cdots & \frac{\partial f_2}{\partial x_n}(z) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(z) & \frac{\partial f_m}{\partial x_2}(z) & \cdots & \frac{\partial f_m}{\partial x_n}(z) \end{bmatrix} = \begin{bmatrix} \nabla f_1(z)^T \\ \nabla f_2(z)^T \\ \vdots \\ \nabla f_m(z)^T \end{bmatrix}$$

- $Df(z)$ is called the *derivative matrix* or *Jacobian matrix* of f at z
- $\widehat{f}$ is a local affine approximation of f around z

Example

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} e^{2x_1+x_2} - x_1 \\ x_1^2 - x_2 \end{bmatrix}$$

- derivative matrix

$$Df(x) = \begin{bmatrix} 2e^{2x_1+x_2} - 1 & e^{2x_1+x_2} \\ 2x_1 & -1 \end{bmatrix}$$

- first order approximation of f around $z = 0$:

$$\widehat{f}(x) = \begin{bmatrix} \widehat{f}_1(x) \\ \widehat{f}_2(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Outline

- notation and terminology
- matrix operations
- linear and affine functions
- **complexity**

Matrix–vector product

matrix–vector multiplication of $m \times n$ matrix A and n-vector x:

$$y = Ax$$

requires $(2n - 1)m$ flops

- m elements in y; each element requires an inner product of length n
- approximately $2mn$ for large n

Special cases: flop count is lower for structured matrices

- A diagonal: n flops
- A lower triangular: n^2 flops
- A sparse: #flops $\ll 2mn$

Matrix–matrix product

product of $m \times n$ matrix A and $n \times p$ matrix B:

$$C = AB$$

requires $mp(2n-1)$ flops

- mp elements in C; each element requires an inner product of length n
- approximately $2mnp$ for large n

Exercises

1. evaluate $y = ABx$ two ways (A and B are $n \times n$, x is a vector)

- $y = (AB)x$ (first make product $C = AB$, then multiply C with x)
- $y = A(Bx)$ (first make product $y = Bx$, then multiply A with y)

both methods give the same answer, but which method is faster?

2. evaluate $y = (I + uv^T)x$ where u, v, x are n-vectors

- $A = I + uv^T$ followed by $y = Ax$

  in MATLAB: `y = (eye(n) + u*v') * x`

- $w = (v^T x)u$ followed by $y = x + w$

  in MATLAB: `y = x + (v'*x) * u`